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# DE MINIMIS AND EQUITY IN RISK

ABSTRACT. Indices and orderings are developed for evaluating alternative strategies in the management of risk. They reflect the goals of reducing individual and collective risks, of increasing equity, and of assigning priority to the reduction and to the equity of high risks. Individual risk is defined as the (random or non-random) level of exposure to a danger. In particular the role of a lower negligibility level is investigated. A class of indices is proposed which involves two parameters, a negligibility level and a parameter of inequality aversion, and several interpretations of the indices are discussed. We provide a set of eight axioms which are necessary and sufficient for this class of indices, and we present an approach to deal with partial information on the parameters.

KEY WORDS: Risk management, social welfare axioms, Harsanyi's view, partial information, ordering of risk distributions.

## 1. INTRODUCTION

Many risks affect a large part of society: the threat of an earthquake or a flood, of an epidemic disease or a new medical drug, the disposal of chemical or nuclear waste, the air pollution caused by an industrial production. To reduce such risks to the environment, public health and safety, various strategies have been developed: e.g., moving the production, constructing a higher smokestack, or installing a filter system. Every such strategy distributes risk in a particular way among the population. Under a given strategy, each individual incurs a certain risk which may be sure or may depend on chance. Thus, the strategy yields a distribution of risks – random or not – in the population.

When we choose between alternative strategies for risk reduction, social and ethical values become relevant. Goals have to be identified which reflect certain ethical values about the distribution of risk in society. For a broader introduction to these problems and for many application examples, the reader is referred to Raiffa (1982).

Here we consider the risk of exposure to a danger, e.g. a toxic air pollutant, where damage to individual health is a function of the level of exposure, say the amount inhaled.

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In risk analysis the notion of risk is defined and made operational in various ways; see Vlek and Stallen (1981). If a person is exposed to risk at some level for sure, this level may be defined as the individual risk, or a function of it, such as a resulting damage to health or an equivalent monetary loss. If the level of exposure is random, risk may be defined as its probability distribution or as a parameter of this distribution: the probability of being exposed at all, the maximum possible level of exposure, the expected exposure, the variance or the upper semivariance of exposure (or of some function of it), etcetera.

For the integration of social and ethical values, several principles and goals have been formulated (Døderlein 1987, ICRP 1991, and others). The first goal is always the reduction of the individual risks. Døderlein (1987) advocates three other goals: risk democracy, risk-cost effectiveness and risk-benefit equity. The three goals imply that the sum of risks and the disparity of risks should both be minimized, which usually conflict with each other. The International Commission on Radiological Protection (ICRP 1991) sets two additional goals focusing on high individual risks: high risks should be reduced and equalized with priority.

In this paper special indices and orderings are developed which reflect these goals. Individual risk is defined as the level of exposure – random or not – which a person receives. (Note that this use of the word risk differs from common use in statistics.) In particular the role of a lower negligibility level is investigated, a two-parameter index is introduced, and an approach is proposed to deal with partial information on the parameters.

For the social evaluation of alternative distributions of individual risks we employ the following postulates.

- **Impartiality:** The evaluation of alternative strategies is based on the distribution of individual risks alone. I.e., it is only relevant that the individuals in the population receive certain risks, but it is irrelevant which particular person receives which risk.

- **De minimis:** There is a threshold level such that all risk levels below the threshold are irrelevant in evaluating alternative strategies.

– **Individualism:** If, with an alternative strategy, a person is better off while the others remain unchanged, then this strategy is preferred.

– **Upper reduction:** A strategy which reduces the risks of persons at higher risk is preferred over one which yields the same total reduction of lower risks.

– **Equity proneness:** If an alternative strategy yields a more equal distribution of risks while the total risk remains unchanged, then this strategy is preferred.

– **Upper equity proneness:** If a strategy increases the equity among persons at higher risks and another strategy yields the same increase of equity among persons at lower risks, then the first strategy is preferred.

Upper equity proneness implies upper reduction; see e.g. Gollier (1993). According to these postulates and more, eight axioms are given in Section 3. From the axioms we will derive a two-parameter class of real-valued indices to evaluate alternative strategies.

Let the population consist of $n$ persons receiving risks, which may be sure or random. We restrict our notation to random risks because non-random risks may be seen as a special case of random ones. Let $X_i$ denote the random level of exposure received by person $i$, and let $F_i$ be its probability distribution function, $i = 1, \ldots, n$. Consider the following two-stage experiment: first, a person is drawn at random and, second, his or her level of risk is observed. Let $Y$ denote the observed result, and $F$ its distribution function, which we will call a *risk distribution function*. In the case of non-random levels, $F(y)$ is the proportion of individuals who receive level $x$ or less. In the case of random levels, $F(y)$ is the probability that a randomly chosen individual is exposed to level $y$ or less.

The social evaluation of different risk distributions $F$ shall be based on an evaluation index, $\varphi : F \mapsto \varphi(F)$. The index represents the preference between distributions by assigning a real number to each of them. Of two given distributions, one is preferred over the other if and only if the index of the first is lower than the index of the second. We will construct such an index from general properties of the underlying social preference order, or, if this is not possible, we will search for an ordering in the set of all distribution functions on $\mathbb{R}_+$ – not necessarily a complete one – which reflects the social preference order by implying it.

Many evaluation indices are the expected value of a non-negative function $h$ which is defined on exposure levels,

$$(1) \qquad \varphi(F) = \int_{[0,\infty[} h(y)dF(y).$$

Equation (1) and the following analysis refer to general[1] distribution functions $F$. In the case of non-random risks $x_i$, $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, $F$ amounts to the empirical probability distribution which gives probability $1/n$ to every $x_1, x_2, \ldots, x_n$, and the integral (1) becomes

$$(2) \qquad \varphi(F) = \frac{1}{n}\sum_{i=1}^{n} h(x_i).$$

We offer three interpretations of the index $\varphi(F)$, subjective expected disutility, utilitarian social illfare, and total detriment.

First, $\varphi(F)$ in Equations (1) and (2) is interpreted as a *subjective expected disutility*: A subject faces the empirical distribution at $x_1, x_2, \ldots, x_n$ without knowing his or her position $i$ in it, i.e. not knowing the risk which will affect him or her. Then $h(x)$ is the subject's *individual evaluation* of being exposed to the level $x$ where individual evaluation is meant either in the von Neumann–Morgenstern sense (under risk, if there are objective probabilities) or in the Savage sense (under uncertainty, if there are only subjective ones). $\varphi(F)$ in Equation (2) is the expected value of the individual evaluation. This view has been introduced by Harsanyi (1953): The subject observes the distribution 'behind a veil'. $\varphi$ is then called a *subjective expected illfare function* which indicates the negative of the expected social welfare in the population. Moreover, the subject who evaluates $(x_1, x_2, \ldots, x_n)$ may think of a nonuniform distribution over the positions $1, 2, \ldots, n$. If $\alpha_i$ is his or her probability of being in the $i$-th position then the evaluation becomes

$$(3) \qquad \varphi(F) = \sum_{i=1}^{n} \alpha_i h(x_i),$$

which is a specialization of Equation (1). More generally, if person $i$ receives a random level $X_i$ for every $i$, Harsanyi's subject faces the overall distribution function $F = \sum_i \alpha_i F_i$, and the subjective evaluation becomes

$$(4) \qquad \varphi(F) = \sum_{i=1}^{n} \alpha_i \int_{[0,\infty[} h(y) dF_i(y),$$

which again is a special case of the index (1).

Second, a different interpretation, *utilitarian social illfare*, can be given to Equation (2) when all levels are non-random; see Kolm (1969), Atkinson (1970), and others. The negative of $h$ is considered as an *individual welfare function*, which is the same for every individual $i$, and a proper utilitarian axiom is imposed on social welfare, e.g. the axiom of 'nonaltruism' in Section 3. Then the social welfare is represented by the sum of all individual welfares, and for its negative we get (2). Equation (3) arises under a modified utilitarian axiom, when persons in different positions $i$ are given different welfare weights $\alpha_i$ which reflect their 'importance' to social welfare.

The third interpretation of Equations (1) and (2), *total detriment*, is given in the concluding section.

A specific feature of most risk reduction policies is the use of a *negligibility level* of exposure. The analysis then focuses on exposures $x$ which exceed this level, say $b$, $b \geqslant 0$. In comparing alternative risk distributions, their left tails up to $x = b$ are considered as negligible. This neglect may be based on one or several of the following reasons:

- the conviction that exposure levels below $b$ are not harmful,
- the fact that they add only insignificantly to existing (and readily accepted) exposure levels,
- the lack of a valid model which quantifies the detriment caused by low levels of exposure,
- the practical difficulty or impossibility of measuring levels of exposure below $b$.

In practical applications, while most experts readily accept the existence of some negligibility level, they often dissent about its numerical value. We shall therefore argue that the de minimis level is included in some interval on which the experts agree and that the evaluation be based on this interval instead of a single value.

The next section presents a specification for the function $h$, and hence the index $\varphi$, which depends on two parameters: negligibility

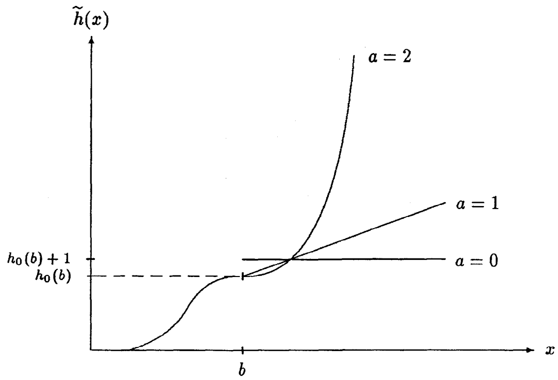


*Fig. 1.* The individual evaluation function $\tilde{h}$ for different parameters $a$.

level and degree of growth. In Section 3 we formulate axioms which reflect the postulates listed above (and more), and we show that they are necessary and sufficient for the proposed class of indices. In Section 4 we assume that we have only partial information about the possible values of the two parameters, *viz.* that they are contained in some known intervals, and we derive decision rules for the comparison of alternative strategies in risk management which are based on the partial information available. Section 5 concludes the paper by relating our approach to cost considerations and sketching its practical implementation in a stepwise decision procedure.

## 2. A TWO-PARAMETER INDEX

We start with a particular specification of $h$. Consider (see Figure 1)

$$\tilde{h}(x) = \begin{cases} h_0(x) & \text{if } 0 \leqslant x \leqslant b, \\ h_0(b) + (x-b)^a & \text{if } x > b, \end{cases} \tag{5}$$

where $a$ and $b$ are parameters $\geqslant 0$, and $h_0(x)$, $0 \leqslant x \leqslant b$, is some function having its maximum at $b$. An axiomatic treatment of this specification and the resulting index $\varphi$ is postponed to Section 3. There we shall show (Theorem 1) that this special choice of $h$ is a consequence of seven axioms, which state the above postulates in a precise way, plus a scaling axiom.

Since $h_0$ in Equation (5) is unknown, we substitute $h_0(x)$ by its upper bound $h_0(b)$, $0 \leqslant x \leqslant b$, and get

$$h(x) = \begin{cases} h_0(b) & \text{if } 0 \leqslant x \leqslant b, \\ h_0(b) + (x-b)^a & \text{if } x > b\,. \end{cases} \tag{6}$$

This results in a ‘conservative’ evaluation of every single distribution. It implies that in comparing distributions the tails below $b$ are irrelevant, which reflects the ‘de minimis’ postulate.

With (6), the evaluation index (1) becomes

$$\varphi(F) = h_0(b) + \int_{]b,\infty[} (y-b)^a dF(y). \tag{7}$$

Basically, $\varphi(F)$ is a truncated upper-$b$ moment.

Let two distribution functions, $F$ and $G$, be given. If we use (7) as an evaluation index, we prefer distribution $G$ *not less* than distribution $F$ if and only if $\varphi(F) \geqslant \varphi(G)$, i.e., if and only if

$$\int_{]b,\infty[} (y-b)^a dF(y) \geqslant \int_{]b,\infty[} (y-b)^a dG(y) \tag{8}$$

holds. The criterion does not depend on the values of $h_0$, which we do not know, but only on the two distributions and the parameters $a$ and $b$.

In the sequel we will analyse the criterion (8) for arbitrary pairs of distribution functions $F$ and $G$. In the empirical distribution case, the criterion (8) reads

$$\sum_{i=i(b)}^{n} (x_i - b)^a \geqslant \sum_{j=j(b)}^{n} (y_j - b)^a$$

where $i(b)$ and $j(b)$ are uniquely determined by $x_{i(b)-1} \leqslant b < x_{i(b)}$ and $y_{j(b)-1} \leqslant b < y_{j(b)}$. The index (7) can be seen as a *weighted sum of excess risks*, i.e. differences between realized risks and the negligibility level,

$$\varphi(F) = h_0(b) + \sum_{i=i(b)}^{n} w_i(x_i - b), \tag{9}$$

where the weights are $w_i = \frac{1}{n}(x_i - b)^{a-1}$. A parameter value $a = 1$ means that the excess risks are equally weighted, while $a > 1$ means increasing weights. When $a = 2$, the excess risks are weighted by themselves. In general, the parameter $a$ indicates the increase in weight when excess risks grow larger. A parameter value $a = 0$ corresponds to the index which is not sensitive to the size of excess risks but only counts the number of persons receiving risks above the negligibility level.

In the non-random levels case, the parameter $a$ indicates the degree of inequity aversion the subject possesses; see Section 3 for details. In the case of random levels, $a$ describes the degree of aversion to the inequality of risk levels which arises in the distribution actually taken by the subject as well as which occurs between individuals in the population.

A principal problem in applying the criterion (8) – or a criterion with a similar function $h$ – consists in the interpretation and the practical assessment of the unknown parameters. A partial information approach to this, which lowers the assessment burden, is given in Section 4.

## 3. AN AXIOMATIC TREATMENT

In this section we restrict our analysis to non-random risks. We provide a set of axioms for the comparison of risk vectors with respect to social illfare and derive the index (7) from them. The axioms refer to the second interpretation, utilitarian social illfare, which was given in the introduction, and reflect the postulates listed there.

A distribution of risk among some number $n$ of persons is described by a *risk vector* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n_+$. The set of all such risk vectors is $D = \bigcup_{n=1}^{\infty} \mathbb{R}^n_+$. We will investigate social evaluation functions $\varphi : D \to \mathbb{R}_+$ and characterize them by proper axioms on the social preference among risk vectors.

Our primitive is a social preference order among risk vectors; it is assumed to be a weak order (transitive and complete) on $D$ which is continuous and essential in at least three components of the risk vectors. A function $\varphi : D \to \mathbb{R}_+$ is said to *represent the social preference* if the following holds: For every $\mathbf{x}$ and $\mathbf{y} \in D$, $\varphi(\mathbf{x}) \leqslant \varphi(\mathbf{y})$ if and only if $\mathbf{x}$ is not less preferred than $\mathbf{y}$. We start with two axioms regarding impartiality and independence of population size.

**(A 1) Impartiality (Symmetry):** If two risk vectors in $\mathbb{R}^n_+$ differ only in that two persons exchange their risks, the risk vectors are equally preferred.

**(A 2) Independence of population size (Replication invariance):** If a risk vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ is compared with its $k$-replication, i.e. with the risk vector $\mathbf{x}^{(k)} = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_n, \ldots, x_n) \in \mathbb{R}^{kn}_+$, then both are equally preferred.

**A 1** implies that, given a population size $n$, the comparison of two risk vectors is based on the empirical distributions alone, and **A 2** extends this to populations of different size. The next axiom captures the idea that a lower negligibility level exists.

**(A 3) De minimis (Independence of low levels):** There is a level $b$ such that two risk vectors in $\mathbb{R}^n_+$ are equally preferred if they differ only in components whose values are in $[0, b]$.

From **A 3** it follows that, for any social evaluation function $\varphi : D \to \mathbb{R}_+$, its behaviour outside the set $D^* = \bigcup_{n=1}^{\infty} ]b, \infty[^n$ is irrelevant. In the sequel, therefore, only the restriction $\varphi^*$ of $\varphi$ to $D^*$ has to be characterized and constructed. Given $\mathbf{x} \in \mathbb{R}^n_+$, let $n^*$ be the number of components $x_i > b$, and $\mathbf{x}^* = (x_1^*, \ldots, x_{n^*}^*)$ the vector of these components.

**(A 4) Individualism (Monotonicity):** If two risk vectors in $\mathbb{R}^n_+$ differ only in the risk which one person receives and if these risk levels are above $b$, then that risk vector is preferred which yields the lower risk to the person.

This last axiom implies that $\varphi^*$ is strictly decreasing (in each argument). We continue with an axiom which we call nonaltruism because, like individualism, it has a taste of lacking altruism: risk changes which concern two persons only are evaluated no matter how well or badly the remaining persons are off.

**(A 5) Nonaltruism (Separability):** If two alternative risk vectors in $\mathbb{R}^n_+$ differ in the risks of only two persons then the preference between them does not depend on the levels of the risks which the remaining persons receive.

In other words, **A 5** says that the conditional social preference of the levels of any two persons is independent of the condition, namely the fixed risk levels of the others.

PROPOSITION 1. *Axioms* **A 1** *to* **A 5** *imply that* $\varphi$ *represents the social preference on* $D$ *if and only if*

$$\varphi(\mathbf{x}) = g\Big(h_0(b) + \frac{1}{n^*}\sum_{i=1}^{n^*} h(x_i^*)\Big), \quad \mathbf{x} \in D, \tag{10}$$

*with some continuous functions* $g$ *and* $h : \mathbb{R}_+ \to \mathbb{R}_+$, $g$ *strictly increasing and* $h$ *strictly increasing on* $[b, \infty[$.

*Proof*. From results of Debreu (1960) and others (see Wakker (1989)) it is well known that the restriction of the preference to $D^*$ has the representation $\varphi^*$,

$$\varphi^*(\mathbf{x}^*) = g^*\Big(\frac{1}{n^*}\sum_{i=1}^{n^*} h^*(x_i^*)\Big), \quad \mathbf{x}^* \in D^*, \tag{11}$$

where $g^* : \mathbb{R}_+ \to \mathbb{R}_+$ and $h^* : ]b, \infty[ \to \mathbb{R}_+$ are strictly increasing continuous functions. By setting $\varphi(\mathbf{x}) = \varphi^*(\mathbf{x}^*)$, $g(\zeta) = g^*(h_0(b) + \zeta)$, and $h(\xi) = h^*(\xi)$ if $\xi > b$, $h(\xi)$ arbitrary otherwise, we get the representation (10). ■

Of course, given some individual illfare function $h$, the index (10) yields the same social preference order for every choice of $g$. Without loss of generality, we may take the identity function for $g$, and thus

$$\varphi(\mathbf{x}) = h_0(b) + \frac{1}{n^*}\sum_{i=1}^{n^*} h(x_i^*). \tag{12}$$

We continue with two axioms on preference for equity: equity proneness and upper equity proneness. They yield further qualitative specifications of $h$, namely that the restriction of $h$ to $]b, \infty[$ is strictly concave and has a derivative which is strictly convex.

**(A 6) Equity proneness:** If two risk vectors in $\mathbb{R}_+^n$ differ only in that two persons receive risks $\xi_1$ and $\xi_2$ with the first and risks $\xi_1 + \varepsilon$ and $\xi_2 - \varepsilon$ with the second risk vector, and there holds $b \leqslant \xi_1 < \xi_1 + \varepsilon < \xi_2 - \varepsilon < \xi_2$, then the second risk vector is preferred.

**(A 7) Upper equity proneness:** If two risk vectors in $\mathbb{R}_+^n$ have the same variance and differ only in that four persons receive risks $\xi_1 < \xi_2 < \xi_3 < \xi_4$ with the first and risks $\xi_1 - \delta$, $\xi_2 + \delta$, $\xi_3 + \varepsilon$, $\xi_4 - \varepsilon$ with the second risk vector and there holds $b \leqslant \xi_1 - \delta < \xi_1 < \xi_2 < \xi_2 + \delta < \xi_3 < \xi_3 + \varepsilon < \xi_4 - \varepsilon$, then the second risk vector is preferred.

Note that the empirical distributions of the two risk vectors in Axiom **A 6** have equal expectations; the same holds in **A 7**. In **A 7** the equal variance assumption is tantamount to saying that $\varepsilon(\xi_2 - \xi_1 + \varepsilon) = \delta(\xi_4 - \xi_3 - \delta)$.

In Axiom **A 6**, the second risk vector is obtained from the first one by a so called *regressive transfer.* A regressive transfer increases equity among the individuals. The opposite of a regressive transfer is called a *progressive transfer*; it decreases equity. In Axiom **A 7**, two transfers are considered simultaneously, a regressive one between persons at high risks and a progressive one having the same size between persons at lower risks. The axiom states that a pair of transfers like these results in a socially preferred distribution. The axioms are well-known in social welfare theory; see e.g. Foster and Shorrocks (1987).

**A 6** (equity proneness) implies that $\varphi^*$ is a strictly Schur-convex function on $D^*$. For Schur-convexity of social evaluation functions, see e.g. Mosler (1994). If $\varphi^*$ has the form (11), **A 6** is equivalent to $h^*$ being strictly convex. Further **A 7** (upper equity proneness) is fulfilled if and only if $h^*$ has a strictly increasing second derivative. We have shown the following proposition:

PROPOSITION 2. *Axioms* **A 1** *to* **A 7** *imply that* $\varphi$ *represents the social preference on* $D$ *if and only if* (10) *holds with* $g$ *strictly increasing and continuous,* $h$ *constant on* $[0, b]$, $h$ *strictly increasing and continuous on* $]b, \infty[$ *and, in addition,* $h$ *strictly concave with a strictly convex derivative on* $]b, \infty[$.

The final axiom, **A 8**, concerns the scaling of the individual risks compared with a properly defined mean risk. Given a risk vector $\mathbf{x} \in D^*$, $\mathbf{x} - b \cdot \mathbf{1}$ is the *excess risk vector;* here $\mathbf{1}$ denotes a vector of ones. The $h$-mean excess risk of $\mathbf{x} \in D$ is defined by

$$\mu_h(\mathbf{x}) = h^{-1}\left(\frac{1}{n^*}\sum_{i=1}^{n^*} h(x_i^*)\right) - b.$$

$\mu_h(\mathbf{x})$ indicates the constant excess risk level which, given to every person, is equally preferred as the risk vector $\mathbf{x}$. Axiom **A 8** postulates that the $h$-mean excess risk is measured on the same scale as the individual excess risks:

**(A 8) Scale (Homogeneity):** For $\mathbf{x} \in D^*$, the $h$-mean excess risk is first degree homogeneous in the excess risk vector, i.e.

$$\mu_h(b \cdot \mathbf{1} + \lambda(\mathbf{x} - b \cdot \mathbf{1})) = \lambda \mu_h(\mathbf{x}) \quad \text{for every } \lambda > 0.$$

When **A 8** is added to the representation (12) of the index, then either

$$h(\xi) = \alpha(\xi - b)^a + \beta \tag{13}$$

or

$$h(\xi) = \alpha \ln(\xi - b) + \beta \tag{14}$$

must hold with some $\alpha > 0, \beta \geqslant 0$. As the preference between risk vectors does not depend on the choice of $\alpha$ and $\beta$, let $\alpha = 1$ and $\beta = 0$. **A 3** implies that $a > 0$, **A 5** that $a > 1$, and **A 6** that $a > 2$. With the full set of axioms we finally get:

THEOREM 1. *Axioms* **A 1** *to* **A 8** *imply that* $\varphi$ *is a representation of the social preference if and only if*

$$\varphi(\mathbf{x}) = h_0(b) + \frac{1}{n^*} \sum_{i=1}^{n^*} (x_i^* - b)^a, \quad \mathbf{x} \in D, \quad \textit{with} \quad a > 2, \quad \textit{or} \tag{15}$$

$$\varphi(\mathbf{x}) = h_0(b) + \frac{1}{n^*} \sum_{i=1}^{n^*} \ln(x_i^* - b), \quad \mathbf{x} \in D, \tag{16}$$

*or a strictly increasing transform of* (15) *or* (16).

*If* **A 7** *(***A 7** *and* **A 8***) are dropped, we have the same representations with* $a > 1$ *(respectively* $a > 0$*) in* (15).

*On the reverse, the representations* (15) *and* (16) *imply the axioms* **A 1** *to* **A 8**.

The necessity of the axioms is easily verified. Equation (15) is the index (7) for empirical distributions.

## 4. PARTIAL INFORMATION ON THE PARAMETERS

In this section we return to the general case of random risks. We assume that there is no unique value of the parameters $a$ and $b$ in the criterion (8), but rather some partial information on their possible values: $a$ is contained in a set $A$, and $b$ in a set $B$, $A$ and $B$ being known. This may occur with a single evaluator who has a limited knowledge about $a$ and $b$ or with a multitude of evaluators who disagree about the correct values.

Consider a given set of alternative strategies each of which results in a distribution $F$. From this set, a choice should be made according to the criterion (8). However, as we have only partial information on the parameters, Equation (7) does not assign a single number to a given distribution function. Therefore we cannot apply the criterion (8) as it stands. What can be done in a first step of the analysis is to remove those strategies from the set of alternatives which are dominated by a remaining alternative in the following sense.

DEFINITION. A distribution function $F$ is $(A, B)$-*dominated* by another distribution function $G$, in symbols $F \leqslant_{A,B} G$, if condition (8) holds for every choice of the parameters $a$ in $A$ and $b$ in $B$, i.e., if $G$ is not less preferred than $F$ for every $a$ and $b$ under consideration.

It is obvious from the definition that $\leqslant_{A,B}$ is a reflexive and transitive relation in the set of all distribution functions on the nonnegative reals, hence a preorder. Of course, the preorder is not complete such that applying it to a given set of alternatives will result in an efficient subset of nondominated alternatives which in general contains more than one element.

As all feasible $a$ are nonnegative, the integrands in (8) are non-decreasing functions. It follows that the preorder is consistent with first degree stochastic dominance (FSD), i.e., if $F$ dominates $G$ in FSD then $F \leqslant_{A,B} G$ for arbitrary sets $A$ and $B$. If $A \subset [1, \infty[$, the integrands are also convex and the preorder is consistent with second degree stochastic dominance in the increasing convex sense ($SSD_{iconv}$); further, if $A \subset [2, \infty[$, the preorder is consistent with third degree stochastic dominance ($TSD_{iconv}$).[2] The following proposition collects the results.

PROPOSITION 3. *Let* $A, B \subset \mathbb{R}_+$.
(i) *If* $F$ *dominates* $G$ *in* FSD *then* $F \leqslant_{A,B} G$.
(ii) *If* $A \subset [1, \infty[$ *and* $F$ *dominates* $G$ *in* $SSD_{iconv}$ *then* $F \leqslant_{A,B} G$.
(iii) *If* $A \subset [2, \infty[$ *and* $F$ *dominates* $G$ *in* $TSD_{iconv}$ *then* $F \leqslant_{A,B} G$.

In case (ii) it follows that the preorder is in accordance with Goal No. 2 of the ICRP which postulates that the dispersion of the individual exposure levels should be reduced (Schneider *et al*. 1993); it follows as well that the highest individual levels of exposure are reduced with priority, which is one possible interpretation of Goal No. 3 of the ICRP; see Gollier (1993). In case (iii), the preorder is consistent with third degree stochastic dominance, which is another interpretation of Goal No. 3; see Schneider *et al*. (1993).

Next we discuss cases where $a$ is uniquely known but $b$ only known to be in an interval. We start with the two special cases $A = \{0\}$ and $A = \{1\}$.

PROPOSITION 4. $F \leqslant_{\{0\},B} G$ *if and only if* $1 - F(b) \geqslant 1 - G(b)$ *for all* $b \in B$ *holds.*

*Proof*. When $a = 0$, (8) becomes $\int_{]b,\infty[} dF(x) - \int_{]b,\infty[} dG(x) \geqslant 0$, i.e., $1 - F(b) \geqslant 1 - G(b)$. ∎

Figure 2 gives an example of two distributions $F$ and $G$ where $F \geqslant_{\{0\},B} G$ and $B$ is an interval.

PROPOSITION 5. $F \leqslant_{\{1\},B} G$ *if and only if* $\int_b^\infty (1 - F(x))dx \geqslant \int_b^\infty (1 - G(x))dx$ *for all* $b \in B$ *holds.*

*Proof*. See the general case in Proposition 6.. ∎

REMARKS. (1) The conditions in Propositions 4. and 5. can be easily checked for given distribution functions $F$ and $G$.

(2) Consider $B = \mathbb{R}_+$ in the the two propositions. It is obvious that $(\{0\}, \mathbb{R}_+)$-dominance coincides with first degree stochastic dominance (FSD), and that $(\{1\}, \mathbb{R}_+)$-dominance coincides with second degree stochastic dominance ($SSD_{iconv}$). The latter is the same as the *generalized Lorenz order*; see e.g. Mosler (1994). When the expected values of $F$ and $G$ are equal, we get the usual Lorenz order.

Proposition 6 will characterize the $(\{a\}, B)$-dominance relation for all integer values of parameter $a$ and arbitrary $B \subset \mathbb{R}_+$. For this,

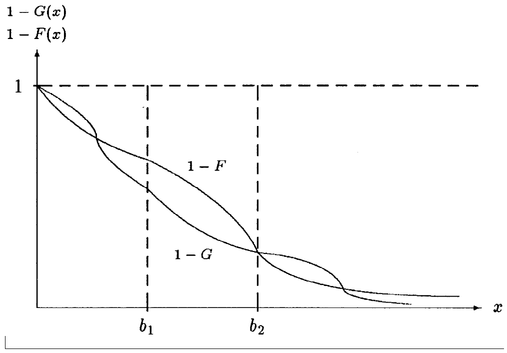


*Fig. 2.* $B = [b_1, b_2],\ F \geqslant_{\{0\},B} G.$

we introduce the notation

$$\overline{F}_1(x) = 1 - F(x), \quad \overline{F}_{a+1}(x) = \int_x^\infty \overline{F}_a(y)dy,$$
$$x \in \mathbb{R}_+, \quad a = 1, 2, 3, \ldots .$$

PROPOSITION 6. *Let* $a \in \{0, 1, 2, \ldots\}$.
*Then* $F \leqslant_{\{a\},B} G$ *if and only if* $\overline{F}_{a+1}(b) \geqslant \overline{G}_{a+1}(b)$ *for all* $b \in B$ .

*Proof.* See the Appendix.

REMARKS. (1) Again, when $B = \mathbb{R}_+$, $(\{a\}, \mathbb{R}_+)$-dominance is equivalent to stochastic dominance of degree $a + 1$, for every $a \in \{0, 1, 2, \ldots\}$. See Rolski (1976) for stochastic dominance of arbitrary integer degree.

(2) Also for non-integer values of $a$, $(\{a\}, \mathbb{R}_+)$-dominance is equivalent to stochastic dominance of degree $a+1$. Stochastic dominance relations of non-integer degree have been investigated by Fishburn (1976, 1980).

(3) For non-integer $a$ there exists no result similar to Proposition 6.

## 5. CONCLUSIONS

The above analysis has developed a class of two-parameter indices for evaluating and comparing alternative strategies in risk management. Their relations to the postulates of anonymity, de minimis, individualism, equity proneness and upper equity proneness have been investigated, and decision rules under partial information have been provided.

For non-random individual risks, we have derived the indices from axioms on the social preference order. This refers to the interpretation of the index as a utilitarian social illfare function. For the other interpretation as a subjective expected disutility, a similar set of axioms can be provided along the lines of the Arrow–Pratt theory of decision under risk.

We should mention that the index (1) in connection with (6) allows for a third interpretation, *total detriment* caused by the risk distribution $F$, as follows. $h(x)$ is understood as the detriment of exposure level $x$ per unit of the population, and $b$ is a level below which no detriment is taken into account; $a$ denotes a shape parameter indicating the velocity of detriment growth beyond level $b$, and $h_0$ is the (unknown) detriment function at low levels. $h_0$ is assumed to be nondecreasing, or at least bounded by its value at $b$. Then $\varphi(F)$ is the total detriment in the population, measured in physical or monetary units, such as the total reduction in expected remaining life or a monetary value assigned to this reduction.

In particular, the specification of $\widetilde{h}$ in Equation (5) is similar to a customary specification of the average cost of detriment caused by a risk level $x$, the so called *monetary man-sievert* multiplied by the level of risk: $\widehat{h}(x) = \alpha_0 x$ if $x \leqslant b$, $\widehat{h}(x) = \alpha_0 (x/b)^a$ if $x > b$; see Schneider *et al*. (1993).

Comparing alternative strategies with respect to total detriment implies that the three postulates of anonymity, de minimis and individualism are satisfied, and, in addition a postulate of risk-cost effectiveness. However, the two equity postulates are neglected.

As our aim is to include ‘equity’ and ‘upper equity’ as goals, we have used a different approach in this paper. We have based the analysis on a subjective evaluation index which can be regarded as a social illfare index and which reflects the postulates of equity and upper equity proneness.

The problem remains how cost considerations can enter this analysis. Any benefit-cost analysis affords that the benefit is measured on an interval scale. But, other than monetary or physical loss, the above subjective evaluation and, in general, social welfare are essentially ordinal concepts. In fact, when integrating a social welfare approach in a decision between alternative risk management strategies, we end up with a two-criteria decision problem, the criteria being costs and the value of the social welfare index. The literature on multi-criteria decision analysis offers a rich methodology to cope with such problems; see e.g. von Winterfeldt and Edwards (1986).

For the problem at hand we propose the following practical approach which is a decision process in steps:

1. Generating the set of all possible alternatives.
2. Removing those alternatives from the set which are either not technically feasible or create costs beyond 'reasonable limits'.
3. Performing the above social welfare evaluation under partial information, which results in a set of non-dominated alternatives.
4. Choosing from this set of alternatives according to their costs and – possibly – according to additional cost-benefit considerations.

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## APPENDIX

*Proof of Proposition 6.* For $a = 0$ see Proposition 4. For $a \geqslant 1$, the proof follows a standard argument of the stochastic dominance literature; see e.g. Fishburn (1976): Observe from the definition of $\overline{F}_a$ that $\overline{F}_a(\infty) = 0$ and $\frac{d}{dx}\overline{F}_{a+1}(x) = -\overline{F}_a(x)$, for every $a \in \mathbb{N}$. By repeated partial integration we derive

$$\int_{]b,\infty[}(y-b)^a dF(y) = -\Big[(y-b)^a(1-F(y))\Big]_b^\infty + \\ + a\int_b^\infty (y-b)^{a-1}(1-F(y))dy$$

$$\begin{aligned}&= -a\Big[(y-b)^{a-1}\overline{F}_2(y)\Big]_b^\infty \\ &\quad + a(a-1)\int_b^\infty (y-b)^{a-2}\overline{F}_2(y)dy \\ &= \dots \\ &= a(a-1)\cdot \ldots \cdot 1 \cdot \int_b^\infty \overline{F}_a(x)dx \\ &= a\,!\ \overline{F}_{a+1}(b)\end{aligned}$$

where all boundary terms are equal to zero. Similarly we get

$$\int_{]b,\infty[}(x-b)^a dG(x) = a\,!\ \overline{G}_{a+1}(b).$$

Hence the inequality (8) is fulfilled if and only if

$$\overline{F}_{a+1}(b) \geqslant \overline{G}_{a+1}(b).$$

When $b$ varies in $B$, the proposition follows. ■

## NOTES

[1] As $h$ is nonnegative, the integral always exists but may equal plus infinity.

[2] We say that $F$ dominates $G$ in $SSD_{iconv}$ if $\int_{\mathbb{R}_+} h(y)dF(y) \geqslant \int_{\mathbb{R}_+} h(y)dG(y)$ for every real function $h$ which is nonnegative, nondecreasing and convex; $F$ dominates $G$ in $TSD_{iconv}$ if the same inequality holds for every $h$ which has, in addition, a nondecreasing second derivative.

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