An interactive decision procedure with multiple attributes under risk

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Consider a finite set of alternatives under risk which have multiple attributes. MARPI is an interactive computer-based procedure to find an efficient choice in the sense of linear expected utility. The choice is based on incomplete information about the decision maker's preferences which is elicited and processed in a sequential way. The information includes qualitative properties of the multivariate utility function such as monotonicity, risk aversion, and separability. Further, in case of an additively separable utility function, bounds on the scaling constants are elicited, and preferences (not necessarily indifferences) between sure amounts and lotteries are asked from the decision maker. The lotteries are Bernoulli lotteries generated by MARPI using special strategies. At every stage of the procedure the efficient set of alternatives is determined with respect to the information elicited so far.

The procedure has been fully implemented on a PC. The paper exhibits the basic ideas of MARPI and some details of its implementation.

Keywords: Partial information, linear expected utility, MAUT, bounds on expected utility, multivariate stochastic dominance (first and second degree, weak and strong), computer-based procedure.

1. Introduction

In prescriptive decision analysis, *Multiple Attribute Utility Theory* (*MAUT*) is a widely used framework to choose a most preferred alternative under risk when there are several decision relevant attributes; see Keeney and Raiffa [9], von Winterfeldt and Edwards [22], and the recent bibliography by Corner and Kirkwood [3]. MAUT is based on linear expected utility (LEU) and on decomposition of the multivariate utility function into sums and products of univariate functions. It provides not only a theoretically sound foundation of choice but also a practical procedure to assess the utility function. Therefore the acronym MAUT is also spelled as Multiple Attribute Utility *Technique*. The classical MAUT procedure, see Keeney and Raiffa [9], first assesses utility independence of the attributes and then elicits the univariate functions and weights from the decision maker by using the certainty equivalence method.

Objections against MAUT have originated from descriptive decision analysis and from practical implementation problems. In descriptive decision analysis, there are the well-known objections against the LEU paradigm arising from observed behaviour, like the Allais paradox and observed ambiguity aversion. As our goal is decision support, we adopt the prescriptive view and restrict ourselves to practical situations where some version of LEU axioms has been accepted by the decision maker. (We believe that these situations are not null.) Further, if a more general model than LEU is assumed which allows, e.g., for ambiguity aversion, under decomposability of the utility function it may reduce to LEU. Dyckerhoff [4] has proved this for the expected utility model with non-additive probabilities when the attributes satisfy the marginality condition.

Besides that, objections against MAUT spring from the difficulties of implementing the procedure to real world situations. These are the main obstacles which often prevent the use of the classical procedure:

(i) A decision analyst is needed to guide and to train the decision maker.

(ii) The procedure can be rather lengthy and expect too much of the decision maker's willingness and ability to respond. If the procedure is stopped before the utility function has been completely assessed the whole effort might be in vain.

(iii) MAUT works only if the attributes show certain kinds of utility independence. But information on this is difficult to assess.

(iv) The procedure urges the decision maker to reveal indifferences between sure amounts and lotteries. However, those responses are often biased and give rise to inconsistencies, see Hershey et al. [6], Hershey and Schoemaker [7], von Nitzsch and Weber [18].

This paper presents an interactive sequential procedure which avoids some of these obstacles. It is called *MARPI* for *Multiattribute Analysis under Risk and Partial Information*. Ad (i), a PC-based decision support system substitutes the decision analyst. The system is menu-driven; it elicits all necessary information from the decision maker in an automatic way. Ad (ii), MARPI can be stopped at any time. Then it gives the set of alternatives which is efficient with respect to the information elicited so far; i.e., this information is used to eliminate inferior alternatives. Ad (iii), before introducing a decomposition, MARPI assesses qualitative properties of the utility function – such as monotonicity, concavity, and others – and checks whether some alternative is stochastically dominated by another with respect to all utility functions having this property. By this, inefficient alternatives are identified and eliminated. Ad (iv), in a second phase MARPI assumes an additive decomposition and faces the decision maker with binary choices between a sure amount and a Bernoulli lottery. (A Bernoulli lottery is a lottery with just two outcomes.) The decision maker is asked for a preferential decision only and *not* urged to find a certainty equivalent (CE) or probability equivalent (PE) to a given lottery. Thus, the biases connected with CE and PE methods do not occur.

There is a vast literature (Aksoy [1]) on interactive multicriteria decision making, but only a few publications on partial information and interactive assessment in MAUT under risk. Sarin [19, 20] introduces a sequential utility bounding procedure in an additive utility model, Weber [21] gives a general framework for using partial information, and Mosler [14, 15] presents stochastic dominance rules to determine an efficient set. Interactive computer programs can be found in Keeney and Sicherman [10], Kirkwood and van der Feltz [11]. For a comparison of computer programs, see Lotfi and Teich [12]. Some related literature concerns MAUT decisions under certainty; see Jacquet-Lagrèze and Siskos [8], who employ holistic judgements, and also the collection by Bana e Costa [2].

Section 2 of the paper exhibits the basic concepts of MARPI, in particular the concepts of partial information employed in Phase One. Section 3 presents the details of Phase Two in which bounds on the expected utility of alternatives are determined. Remarks on the implementation are found in section 4 while section 5 concludes the paper.

2. Basic concepts

We start with a set $\mathscr{A}$ of decision alternatives which are characterized by the levels of n relevant attributes, $n \geq 1$. For each alternative there is a number of possible outcome vectors x in $\mathbb{R}^n$. Thus, an alternative is described by a random vector in n-space. We assume that the probability distributions of all alternatives are known (as either subjective or objective probabilities), i.e., we assume a situation of decision making under risk.

The procedure MARPI processes the set $\mathscr{A}$ of alternatives in a sequential way. In each step, the decision maker is asked for some information on her preferences. MARPI uses the information elicited so far to eliminate inferior alternatives and presents the actual efficient set to the decision maker. The procedure stops when the decision maker wants it to stop or when the actual efficient set becomes a singleton.

More formally, let $\mathscr{A}$ be a finite set of probability distribution functions in $\mathbb{R}^n$. Let their joint support be finite, and let S be a rectangle containing it. We assume that the decision maker has a preference relation $\succeq$ on the set of lotteries on S and that she accepts some version of LEU axioms. Then, of two alternatives F and G in $\mathscr{A}$, F is not less preferred than G, $F \succeq G$, if

$$\int u(x)\,\mathrm{d}F(x) \geq \int u(x)\,\mathrm{d}G(x), \tag{1}$$

where u is the decision maker's n-variate utility function. For a vector $x \in \mathbb{R}^n$ and $J \subset \{1,2,\dots n\}$, let x_{-J} denote the vector where all components with index $i \in J$

have been cancelled. E.g., $x = (2, 5.5, 7, 0)$, $J = \{1, 3\}$, $x_{-J} = (5.5, 0)$. Let $\succeq_J (x_{-J})$ denote the conditional preference relation over lotteries where the levels of all attributes $i \notin J$ are non-random and fixed at x_{-J}. For univariate conditional preferences we write $\succeq_i (x_{-i})$ instead of $\succeq_{\{i\}} (x_{-\{i\}})$.

When the procedure starts neither $\succeq$ nor u are known. A given set U of n-variate functions is called a *partial information* about u. If (1) holds for all $u \in U$ we say that *F dominates G with respect to U*, $F \succeq_U G$.

Let $\mathscr{A}(0) = \mathscr{A}$. In each step t of the procedure a partial information $U(t)$ is assessed from the decision maker, $U(1) \supset U(2) \supset \ldots$, and a set $\mathscr{A}(t)$ is computed, $\mathscr{A}(t) \subset \mathscr{A}(t-1)$, such that every alternative which is non-dominated with respect to $U(t)$ is an element of $\mathscr{A}(t)$,

$$\mathscr{A}(t) \supset \{G \in \mathscr{A}(t-1) : \text{there is no } F \in \mathscr{A}(t-1) \text{ with } F \neq G \text{ and } F \succeq_{U(t)} G\}.$$

With MARPI, we distinguish two phases. In Phase One the incomplete information applies to general properties like monotonicity of the preference, global and pairwise risk aversion, and utility independence of attributes. If the marginality condition holds, i.e., the utility function has an additive decomposition, a second phase follows.

In the sequel we present instances of partial information which have been implemented in Phase One of MARPI to perform dominance checks. The notions and results are standard. See, e.g., Keeney and Raiffa [9] if not otherwise stated.

2.1. MONOTONICITY OF THE PREFERENCE

First, the decision maker is asked whether for every x_{-i} the conditional preference $\succeq_i (x_{-i})$ is increasing or decreasing, i.e., equivalently, whether u is increasing or decreasing in its ith variable. (Here and in the whole paper, increasing means non-decreasing, and decreasing means non-increasing.) When $\succeq_i (x_{-i})$ is increasing for every x_{-i} and every i we have the partial information

$$\mathscr{U}_{inc} = \{u | u : \mathbb{R}^n \to \mathbb{R},\ u \text{ increasing}\}.$$

Dominance with respect to $\mathscr{U}_{inc}$ is called *strong first degree stochastic dominance*, shortly *strong FSD*. For a necessary and sufficient characterization which can be computationally checked, see the appendix.

When for some i the decision maker responds that $\succeq_i (x_{-i})$ is decreasing, attribute i should be substituted by its negative. When the response is neither increasing nor decreasing, the decision maker is suggested to redefine attribute i. E.g., if the attribute measures the difference between the actual value and some target value she may redefine it as the distance from the target value.

2.2. GLOBAL AND CONDITIONAL RISK AVERSION

The preference is globally *risk averse* (*risk prone*) if no lottery is preferred to (postponed to) its expectation vector, i.e., if the utility function u is concave (resp. convex). Let $\mathscr{U}_{conc}$ and $\mathscr{U}_{conv}$ denote the sets of concave and convex utility functions, respectively; similarly, $\mathscr{U}_{iconc}$ and $\mathscr{U}_{iconv}$ the sets of utility functions which are, in addition, increasing. With respect to these four kinds of partial information, four dominance relations are defined. In MARPI, the decision maker is asked whether she is globally risk averse or prone. Then, she may activate the relevant check under the heading *strong second degree stochastic dominance*, shortly *strong SSD*. The preference is called *risk averse* (*risk prone*) *in the ith attribute* if this holds for the conditional preference, i.e., the utility function is concave (resp. convex) in its ith argument.

2.3. PAIRWISE AND n-VARIATE RISK AVERSION

Let the preference be increasing, $n = 2$, and consider a lottery L giving the pairs (a_1, a_2) and (b_1, b_2) in $\mathbb{R}^2$, each with probability 0.5, and another lottery M giving (a_1, b_2) and (b_1, a_2) with equal probabilities. If M is more (less) preferred than L for all $a_1 \le b_1$ and $a_2 \le b_2$, then the preference is called *pairwise risk averse* (*pairwise risk prone*). Equivalently, if u is differentiable, we have $\partial^2 u/\partial x_1 \partial x_2 \le 0$ (≥ 0 resp.). The notion has nothing to do with global or conditional risk aversion as mentioned above. Pairwise risk aversion may, in particular, occur when the two attributes are consumption levels of substitutional goods. For general $n \ge 2$, an increasing preference is named *pairwise risk averse* (*pairwise risk prone*) if the bivariate conditional preferences have this property, i.e., $\partial^2 u/\partial x_i \partial x_j \le 0$ for all $i \ne j$. Further, under differentiability, the preference is called *n-variate risk averse* if

$$(-1)^k \frac{\partial^k u}{\partial x_{i_1} \dots \partial x_{i_k}} \le 0 \quad \text{for } 1 \le i_1 < i_2 < \dots i_k \le n,\ k = 1, 2, \dots n.$$

When $n = 2$, this means increasingness and pairwise risk aversion. The preference is called *n-variate risk prone* if

$$\frac{\partial^k u}{\partial x_{i_1} \dots \partial x_{i_k}} \ge 0 \quad \text{for } 1 \le i_1 < i_2 < \dots i_k \le n,\ k = 1, 2, \dots n.$$

Let $\mathscr{G}_1^-$ and $\mathscr{G}_1^+$ denote the sets of all n-variate risk averse and all n-variate risk prone utility functions, respectively. Note that both sets are in $\mathscr{U}_{inc}$. For details the reader is referred to Mosler and Scarsini [16]. E.g., the function u given by

$$u(x_1, x_2, \dots x_n) = \gamma - \delta \prod_{i=1}^{n} (-v_i(x_i)) \tag{2}$$

is in $\mathscr{G}_1^-$ whenever the v_i are increasing real functions, $v_i \leq 0$, and $\delta > 0$. The decision maker may activate dominance checks with respect to $\mathscr{G}_1^-$ and $\mathscr{G}_1^+$.

2.4. MARGINALITY CONDITION

When choices between given lotteries are based on their univariate marginal distributions only, the preference is said to satisfy the *marginality condition*. Then the utility function is *additive separable*. I.e., it has an *additive decomposition*,

$$u(x_1, x_2, \ldots x_n) = \alpha + \beta \sum_{i=1}^{n} u_i(x_i), \tag{3}$$

with $\beta > 0$, and there exists a standardized additive decomposition of an equivalent u,

$$u(x_1, x_2, \ldots x_n) = \sum_{i=1}^{n} \alpha_i u_i(x_i), \tag{4}$$

where the α_i are scaling constants, $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^{n} \alpha_i = 1$, and the u_i are univariate utility functions, $0 \leq u_i \leq 1$. In this case, the preference obviously is *n-variate risk neutral*, i.e., both n-variate risk averse and prone.

2.5. UTILITY INDEPENDENCE

A set J of attributes is said to be *utility independent* if the conditional preference $\succeq_J (x_{-J})$ does not depend on x_{-J}. If $n \geq 3$ and every $J \subset \{1, 2, \ldots n\}$ is utility independent, the utility function has either an additive decomposition (3) or a *multiplicative decomposition*,

$$u(x_1, x_2, \ldots x_n) = \alpha + \beta \prod_{i=1}^{n} u_i(x_i). \tag{5}$$

There holds $u \in \mathscr{G}_1^+$ if u has a multiplicative decomposition (5) with increasing $u_i, u_i \geq 0, \beta > 0$. Let $\mathscr{F}_2^+$ be the set of all u where, in addition, the u_i are convex, i.e., the preference is risk prone in every attribute. Further, let $\mathscr{F}_2^-$ be the set of all u having a multiplicative decomposition (2) with increasing $v_i, v_i \leq 0, \delta > 0$, and in addition, the v_i being concave, i.e., risk averse in the attributes.

During Phase One, MARPI offers the following dominance checks with respect to the partial information mentioned. For the conditions which are computationally checked, see the appendix.

- *strong FSD* w.r.t. $\mathscr{U}_{inc}$.
- *strong SSD* w.r.t. $\mathscr{U}_{conc}$, $\mathscr{U}_{conv}$, $\mathscr{U}_{iconc}$, or $\mathscr{U}_{iconv}$, respectively.
- *weak FSD* w.r.t. $\mathscr{G}_1^-$.

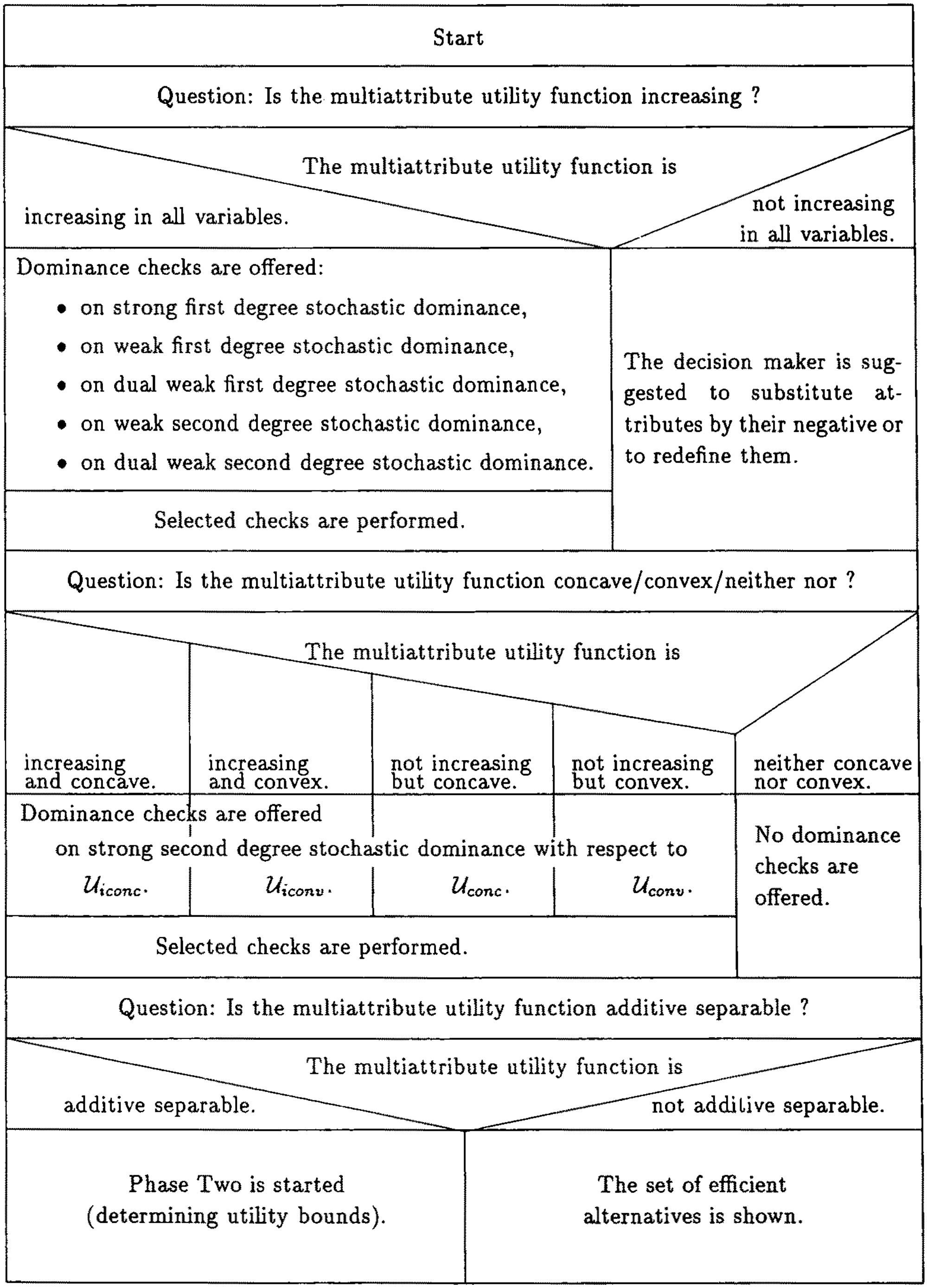

Fig. 1. Phase One of MARPI.

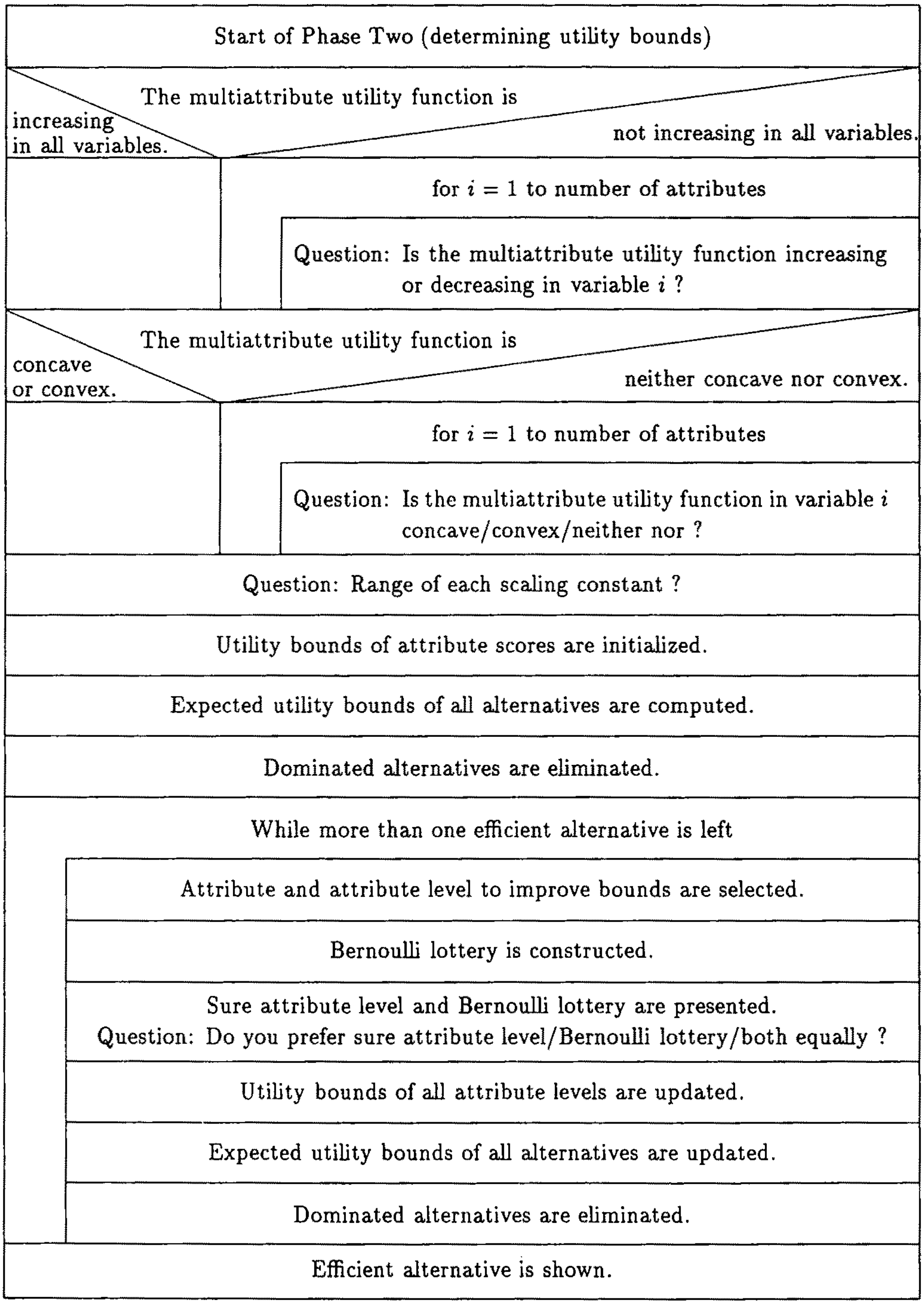

Fig. 2. Phase Two of MARPI.

- *dual weak FSD* w.r.t. $\mathscr{G}_1^+$.
- *weak SSD* w.r.t. $\mathscr{F}_2^-$.
- *dual weak SSD* w.r.t. $\mathscr{F}_2^+$.

The latter two relations are equivalent to dominance with respect to the weaker partial informations $\mathscr{G}_2^+$ and $\mathscr{G}_2^-$ which include $\mathscr{F}_2^+$ and $\mathscr{F}_2^-$, respectively. See Mosler and Scarsini [16].

A structure chart of Phase One is shown in figure 1. At the end of Phase One the decision maker is asked whether the marginality condition or an equivalent condition holds. If this is true, Phase Two is started in which further partial information is assessed and used to reduce the efficient set. Phase Two includes the partial assessment of an additive utility function, and the identification of efficient alternatives with respect to this partial information. A structure chart of Phase Two, the assessment procedure, is shown in figure 2. The basic ideas and some details are described in section 3.

3. Determining utility bounds

In Phase Two of the procedure an additive utility function u of the form (4) is assumed. So, the assessment of u is reduced to the assessment of n univariate utility functions u_i and n scaling constants α_i. The identification of efficient alternatives proceeds as follows:

3.1. SHAPE OF THE UNIVARIATE UTILITY FUNCTIONS

First, the shape of the u_i is asked from the decision maker. Each u_i must be either increasing or decreasing. Additionally, each u_i may be either concave or convex or none or both. If one of these properties has already been elicited for u, it follows for all u_i and the corresponding question is skipped. In each attribute i there is a finite number of possible values ξ. Let $x_i^{\oplus}$ denote the best level of attribute i, $x_i^{\ominus}$ the worst level. We restrict our exposition to the case where all univariate utility functions are increasing, hence $x_i^{\ominus} < x_i^{\oplus}$ for all i. (When some u_i are decreasing the procedure runs in the obviously modified way.)

3.2. DETERMINATION OF THE SCALING CONSTANTS

Next, the decision maker has to give information on the scaling constants α_i. She is not asked to specify the α_i exactly. Required is the input of a range $[\underline{\alpha}_i, \bar{\alpha}_i]$ with $0 \leq \underline{\alpha}_i \leq \bar{\alpha}_i \leq 1$ for each α_i. A response of the decision maker is only accepted if there exists at least one vector $(\alpha_1, \ldots, \alpha_n)$ with $\sum_{i=1}^n \alpha_i = 1$, and $\underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i$ for all i. Support is provided by presenting $x_i^{\ominus}$ and $x_i^{\oplus}$ for all attributes i. Since the ranges $[\underline{\alpha}_i, \bar{\alpha}_i]$ remain unchanged in the whole procedure, the decision maker is urged to choose the bounds $\underline{\alpha}_i$ and $\bar{\alpha}_i$ as tight as possible in order to shorten the following iteration process.

The embedding of an assessment procedure for the α_i in the iteration process is under preparation. The assessment will be based on additional lottery comparisons and will lead to a step by step improvement of the bounds on the α_i.

3.3. INITIALIZATION OF BOUNDS ON THE UTILITY OF ATTRIBUTE LEVELS

At every stage t of the procedure, the elicited information is stored in the form of bounds on the u_i. Let $\underline{u}_i(\xi)$ denote the lower bound on $u_i(\xi)$, and $\bar{u}_i(\xi)$ its upper bound, $i = 1, \dots n$, at stage t.

The bounds are initialized in the following way:

- $\underline{u}_i(x_i^{\oplus}) = 1 = \bar{u}_i(x_i^{\oplus})$ and $\underline{u}_i(x_i^{\ominus}) = 0 = \bar{u}_i(x_i^{\ominus})$ for all i.
- If u_i is neither concave nor convex then $\underline{u}_i(\xi) = 0$ and $\bar{u}_i(\xi) = 1$ for all levels ξ with $x_i^{\ominus} < \xi < x_i^{\oplus}$.
- If u_i is concave then $\underline{u}_i(\xi)$ is obtained by linear interpolation between the $x_i^{\oplus}$ and $x_i^{\ominus}$. So, $\underline{u}_i(\xi) = (\xi - x_i^{\ominus})/(x_i^{\oplus} - x_i^{\ominus})$ and $\bar{u}_i(\xi) = 1$ for all levels ξ with $x_i^{\ominus} < \xi < x_i^{\oplus}$. See figure 3.
- Similarly, if u_i is convex then $\underline{u}_i(\xi) = 0$ and $\bar{u}_i(\xi) = (\xi - x_i^{\ominus})/(x_i^{\oplus} - x_i^{\ominus})$ for all levels ξ with $x_i^{\ominus} < \xi < x_i^{\oplus}$.

3.4. COMPUTATION OF EXPECTED UTILITY BOUNDS

Next, lower and upper bounds on the expected utility of each alternative are established. At stage t, let $\underline{U}_k$ denote our lower bound on the expected utility of alternative k, and $\bar{U}_k$ our upper bound. To obtain $\underline{U}_k$ and $\bar{U}_k$ we solve the following linear programs:

$$\begin{aligned} &\text{Minimize} \quad \underline{U}_k = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m_k} p_{kj}\, \underline{u}_i(x_{ikj}) \\ &\text{subject to} \quad \sum_{i=1}^{n} \alpha_1 = 1, \\ &\qquad\qquad\quad \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i \quad \forall i, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\text{Maximize} \quad \bar{U}_k = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m_k} p_{kj}\, \bar{u}_i(x_{ikj}) \\ &\text{subject to} \quad \sum_{i=1}^{n} \alpha_i = 1, \\ &\qquad\qquad\quad \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i \quad \forall i, \end{aligned} \tag{7}$$

where m_k denotes the number of possible events concerning alternative k, p_{kj} denotes the probability of event j with alternative k, and x_{ikj} denotes the level of attribute i with alternative k if event j occurs.

$\underline{U}_k$ and $\bar{U}_k$ have to be computed for all k in $\mathscr{A}(t)$, the current efficient set. The optimal solutions of the variables α_i in (6) and (7) have no relevance for the further iteration process, because each solution represents only one possible specification of the unknown scaling constants. The above linear programs are solved with the revised simplex method. See, e.g., Murty [17].

3.5. ELIMINATION OF DOMINATED ALTERNATIVES

If $\underline{U}_l \geq \bar{U}_k$ holds at some $k \neq l$, alternative k is dominated by alternative l with respect to the information elicited so far. The program checks this for every pair in $\mathscr{A}(t)$, removes dominated alternatives, and presents the result as a new efficient set $\mathscr{A}(t+1)$.

However, this procedure may be strengthened. Given two alternatives k and l the optimal solutions of (6) and (7) to compute $\underline{U}_l$ and $\bar{U}_k$ may have different values of α_i. But the expected utility evaluation of k and l is based on the same utility function and exactly one of the vectors $(\alpha_1, \ldots, \alpha_n)$ with $\sum_{i=1}^n \alpha_i = 1$ and $\underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i$, $\forall i$, represents the true but unknown scaling constants. So it may happen that $\underline{U}_l < \bar{U}_k$ but $\sum_{i=1}^n \alpha_i \sum_{j=1}^{m_l} p_{lj}\, \underline{u}_i(x_{ilj}) \geq \sum_{i=1}^n \alpha_i \sum_{j=1}^{m_k} p_{kj}\, \bar{u}_i(x_{ikj})$ for all vectors $(\alpha_1, \ldots, \alpha_n)$ with $\sum_{i=1}^n \alpha_1 = 1$ and $\underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i$, $\forall i$. Thus sharper comparisons of bounds can be introduced. This results in the following stronger dominance check: Solve the linear program

$$\begin{aligned} &\text{Minimize} \quad Z_{lk} = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{m_l} p_{lj}\, \underline{u}_i(x_{ilj}) - \sum_{j=1}^{m_k} p_{kj}\, \bar{u}_i(x_{ikj}) \right) \\ &\text{subject to} \quad \sum_{i=1}^{n} \alpha_1 = 1, \\ &\qquad\qquad\quad \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i \quad \forall i, \end{aligned} \tag{8}$$

for all ordered pairs in $\mathscr{A}(t)$. Z_{lk} gives the minimum possible difference between the lower bound on the expected utility of alternative l and the upper bound on the expected utility of alternative k. So, if already $Z_{lk} \geq 0$ holds, alternative k is dominated by alternative l with respect to the elicited information and can be eliminated from $\mathscr{A}(t)$. This yields a set $\mathscr{A}(t+1)$ which is possibly smaller than $\mathscr{A}(t+1)$ defined above.

Let N be the number of elements in $\mathscr{A}(t)$. If we compute $\underline{U}_k$ and $\bar{U}_k$ from (6) and (7) we have to solve $2N$ linear programs. In comparison, if we want to perform the stronger dominance check (8) we need to solve $N(N-1)$ linear programs (8) to check all ordered pairs of efficient alternatives. For computing time reasons, in the present implementation of MARPI we use the weaker dominance check. In a future implementation, those pairs of alternatives for which $(\bar{U}_k - \underline{U}_l)$ is small will be subject to an additional stronger dominance check (8).

If $\mathscr{A}(t+1)$ becomes a singleton the most preferred alternative is identified, and the procedure stops. Otherwise, the decision maker is asked for more information on the u_i. This is done by revealing preferences of the decision maker between a given Bernoulli lottery and a sure amount in attribute i. From this preference judgement improved bounds on the utility of the presented sure amount are derived. In the next two subsections we discuss how the attribute i, the sure amount, and the Bernoulli lottery are selected.

3.6. SELECTION OF THE ATTRIBUTE AND THE SURE AMOUNT TO BE PRESENTED

The improvement of the bounds on the utility of the presented sure amount should result in a relatively strong tightening of $\underline{U}_k$ and $\bar{U}_k$. This is the case for attributes i and levels ξ

which have a high scaling constant α_i,

whose upper and lower utility bounds $\bar{u}_i(\xi)$ and $\underline{u}_i(\xi)$ have a large difference,

and which are occurring in the efficient alternatives with a high probability mass.

To identify such a level we compute for every i and ξ

$$g_i(\xi) = \left(\sum_{k \in \mathscr{A}(t)} \sum_{j=1}^{m_k} p_{kj} 1_{\{x_{ikj}\}}(\xi) \right) (\bar{u}_i(\xi) - \underline{u}_i(\xi)) \left(\frac{\bar{\alpha}_i + \underline{\alpha}_i}{2} \right),$$

and the attribute level ξ^* which maximizes $g_i(\xi^*)$ is chosen for the presentation as a sure amount in the next preference question.

3.7. CONSTRUCTION OF THE LOTTERY TO BE PRESENTED

Given i and ξ^* a Bernoulli lottery L is constructed. Let L yield ξ^+ with probability p and ξ^- with probability $1-p$.

Since at the beginning only the utilities of $x_i^{\oplus}$ and $x_i^{\ominus}$ are exactly known, we use them as ξ^+ and ξ^- in the whole procedure.

We determine p with respect to the present bounds on $u_i(\xi^*)$ such that independently of the decision maker's response to the preference question, a bisection of the difference between $\underline{u}_i(\xi^*)$ and $\bar{u}_i(\xi^*)$ is obtained,

$$p := \frac{\underline{u}_i(\xi^*) + \bar{u}_i(\xi^*)}{2}.$$

An obvious modification of the procedure is the following. Depending on ξ^* and the stage choose some ξ^+ and ξ^- for which $\bar{u}_i(\xi^+) - \underline{u}_i(\xi^+)$ and $\bar{u}_i(\xi^-) - \underline{u}_i(\xi^-)$ are rather small.

3.8. REVISION OF THE UTILITY BOUNDS AT ξ^*

ξ^* and L are shown to the decision maker. She has to state whether she prefers the sure amount or the lottery or whether she is indifferent between both, i.e., whether

$$u_i(\xi^*) \text{ greater, smaller, or equal } \frac{\underline{u}_i(\xi^*) + \bar{u}_i(\xi^*)}{2},$$

respectively. If " $>$" holds we get a new lower bound for $u_i(\xi^*)$, if "$<$ " holds we get a new upper bound. In both cases one bound has been sharpened and the distance between the upper and the lower bound at ξ^* has become half of the distance before. If " $=$ " holds we get an exact value for $u_i(\xi^*)$, and the new upper and lower bounds coincide.

3.9. REVISION OF THE UTILITY BOUNDS OF OTHER ATTRIBUTE LEVELS

Simultaneously to the change of the bounds on $u_i(\xi^*)$ the utility bounds of other attribute levels ξ are improved. If u_i is neither concave nor convex then, based on the monotonicity property of u_i, $\underline{u}_i(\xi)$ and $\bar{u}_i(\xi)$ are revised for all attribute levels ξ^+ with $\xi^* < \xi^+ < x_i^\oplus$ and ξ^- with $x_i^\ominus < \xi^- < \xi^*$ as follows:

$$\underline{u}_i^{\text{new}}(\xi^+) := \max(\underline{u}_i(\xi^*), \underline{u}_i^{\text{old}}(\xi^+)),$$
$$\bar{u}_i^{\text{new}}(\xi^-) := \min(\bar{u}_i(\xi^*), \bar{u}_i^{\text{old}}(\xi^-)).$$

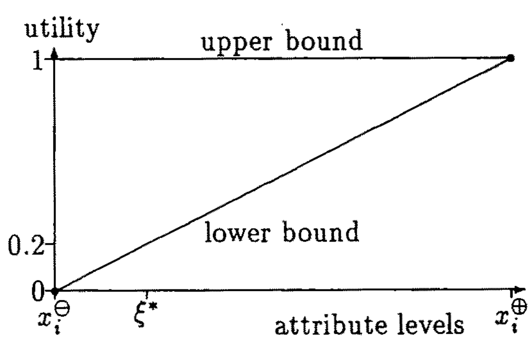

Fig. 3. Initialization of bounds on a concave utility function.

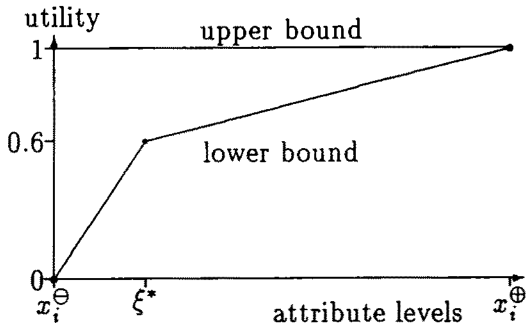

Fig. 4. Improved utility bounds if the decision maker prefers ξ^* over L.

If, in addition, u_i is concave or convex then tighter bounds are obtained. To illustrate this, let us consider the case that u_i is concave. Then the bounds on the utility of the attribute levels of attribute i have been initialized as shown in figure 3. Assume that according to subsection 3.6 the attribute level ξ^* has been selected. Since $\underline{u}_i(\xi^*) = 0.2$ and $\bar{u}_i(\xi^*) = 1$, we compute $p = 0.6$. So, the decision maker is faced with a choice between the sure amount ξ^* and the lottery L yielding $x_i^\oplus$ with probability 0.6 and $x_i^\ominus$ with probability 0.4.

If the decision maker prefers ξ^* over L then we obtain the new lower bound $\underline{u}_i(\xi^*) = 0.6$. Based on the concavity of u_i, $\underline{u}_i(\xi^-)$ is revised for all ξ^- with $x_i^\ominus < \xi^- < \xi^*$ by linear interpolation between $\underline{u}_i(x_i^\ominus)$ and $\underline{u}_i(\xi^*)$. Analogously, $\underline{u}_i(\xi^+)$ is improved for all ξ^+ with $\xi^* < \xi^+ < x_i^\oplus$ by linear interpolation between $\underline{u}_i(\xi^*)$ and $\underline{u}_i(x_i^\oplus)$. The shape of the new bounds is shown in figure 4.

Otherwise, if the decision maker prefers L over ξ^* we get the new upper bound $\bar{u}_i(\xi^*) = 0.6$. From the concavity of u_i we deduce that $\bar{u}_i(\xi^-)$ can be revised for all ξ^- with $x_i^\ominus < \xi^- < \xi^*$ by linear extrapolation of $\bar{u}_i(\xi^*)$ and $\bar{u}_i(x_i^\oplus)$. Analogously, $\bar{u}_i(\xi^+)$ is improved for all ξ^+ with $\xi^* < \xi^+ < x_i^\oplus$ by linear extrapolation of $\bar{u}_i(x_i^\ominus)$ and $\bar{u}_i(\xi^*)$. See figure 5.

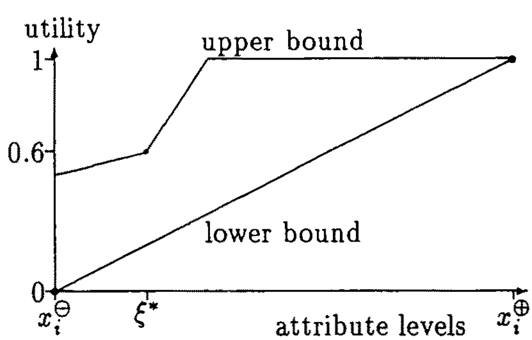

Fig. 5. Improved utility bounds if the decision maker prefers L over ξ^*.

If the decision maker is indifferent between ξ^* and L we get the described improvements of the lower bounds as well as of the upper bounds.

In case of a convex univariate utility function the computations are similar.

After the utility bounds of all levels of attribute i have been revised we proceed with the computation of new expected utility bounds as described in subsection 3.4.

3.10. TERMINATION OF THE BOUNDING PROCEDURE

The assessment procedure stops in the following cases:

(i) The decision maker wants to stop the procedure. Then the current set of efficient alternatives $\mathscr{A}(t)$ is presented.

(ii) $\mathscr{A}(t)$ becomes a singleton. Then, the most preferred alternative has been identified.

(iii) The utilities $u_i(\xi)$ of all attribute levels occurring in $\mathscr{A}(t)$ have been determined exactly. In this case, the expected utility bounds of alternatives cannot be improved further unless the decision maker provides sharper bounds on the scaling constants.

4. Implementation

MARPI has been implemented on a personal computer. The program is written in Borland Pascal 7.0 and can be run on a standard configuration of a PC IBM compatible with at least 1 MByte of RAM memory. The CPU-times depend strongly on the given data. Because the dominance checks need extensive mathematical computations a math coprocessor is recommended. For the solution of large problems with many attributes a 486 DX processor is necessary to keep computing times within tolerable limits.

MARPI is designed as an interactive and event oriented program. The user surface is based on Turbo Vision 2.0 which uses a windowing technique and provides mouse support. The conception of the program enables its usage by a user who has only little computer experience.

The user is assumed to know the linear expected utility model and the basics of MAUT as exhibited in Keeney and Raiffa [9]. But it is not necessary for her to be a specialist in utility theory because at every stage of the procedure a context sensitive information is provided.

The package consists of two modules:

- The first module, named "data", enables the user to input and edit multiattribute decision problems under risk. Moreover, it is possible to save the data on and to load the data from disk. During input and editing the user is supported by consistency checks so that, for example, only valid probability distributions are accepted, and in addition it is ensured that in each alternative no outcome occurs twice or more.

- The second module, named "session", includes the above interactive procedure of identification of the efficient alternatives.

Some further developments of the package are in progress.

5. Conclusions

An interactive procedure has been developed and implemented which helps a decision maker to find a set of efficient alternatives. Various kinds of information are elicited from the decision maker in a sequential way, and at each stage the set of alternatives is determined which is efficient with respect to the information elicited so far.

While this procedure is a general one, in practical applications the parameters should stay within certain limits: a maximum of five attributes, some ten different levels per attribute, and thirty alternatives seem to be feasible. The number of alternatives might be cut down by screening procedures based on satisfaction levels and some hierarchy of attributes. The choice of relevant attributes should be guided by considerations of relative importance and of mutual independence.

If the marginality condition does not hold (i.e., the attributes are not additively independent) the procedure stops after Phase One. There remain two possibilities: Either the decision maker chooses from the efficient set by some other device, e.g., by direct inspection, or she redefines the attributes in order to get new ones which satisfy the marginality condition. Some authors point out that the results of utility function assessment might be rather biased if the marginality condition fails; see von Nitzsch and Weber [18] and others. They argue that whenever preference dependencies between the attributes are revealed the decision maker should be asked to redefine the attributes. Whether redefining the attributes is feasible depends on the practical situation. In particular, the decision maker may have the power to redefine attributes or not, and the necessary information on the redefined attributes may be obtainable or not.

During Phase Two an additive separable utility function is assumed. Most existing procedures are restricted to an additive utility function. In the standard approach, first the univariate utility functions and the scaling constants are specified exactly. This requires a large number of indifference judgements from the decision maker. Second, the expected utilities of all alternatives are computed and compared to identify the optimal alternative. However, the applicability of the standard approach is rather limited. To get practical results, it is necessary to reduce the number of judgements as well as to simplify the type of judgements required from the decision maker. Some authors do this by making parametric assumptions about the shape of the univariate utility functions. So, for instance, exponential univariate utility functions are assumed. But these are strong assumptions which in many practical cases cannot be verified.

The bounding procedure used in MARPI has been developed as a new approach to identify preferred alternatives. For finding efficient choices it is not necessary to determine the utility function exactly. Our procedure is based on the sequential use of partial information on the additive utility function elicited from the decision maker. No indifference judgements are requested but only preference judgements.

A number of further developments of MARPI is in progress. They include other characterizations and interactive presentations of different kinds of information in Phase One, and the determination of scaling constants through lottery comparisons and additional comparisons of expected utility bounds in Phase Two. Likewise, further possibilities of storing and interactively presenting the information assessed and the results computed shall be developed.

Appendix: Characterization of strong and weak FSD and SSD

Strong FSD and SSD can be characterized by Markov kernels. For proofs see the survey on kernel dominance in Mosler and Scarsini [16] and the references given there. Kernel dominance means that F dominates G if and only if

$$P_G(\cdot) = \int M(x,\cdot)\mathrm{d}P_F(x), \tag{9}$$

where P_F and P_G are the probability measures belonging to F and G, and M is some Markov kernel.

When the distributions are finitely discrete, the situation is particularly simple. Let $S = \{s^1, s^2, \ldots s^m\} \subset \mathbb{R}^n$ be the joint support of F and G and consider the Markov kernel M given by

$$M(x,\{y\}) = \begin{cases} \pi_{jk}, & \text{if } x = s^j \text{ and } y = s^k, j,k \in \{1,2,\ldots m\}, \\ 1, & \text{if } x \notin S \text{ and } y = x, \\ 0, & \text{else,} \end{cases}$$

with

$$\pi_{jk} \geq 0 \quad \text{for all } j,k, \tag{10}$$

and

$$\sum_{k=1}^{m} \pi_{jk} = 1 \quad \text{for all } j. \tag{11}$$

Then (9) becomes

$$P_G(\{s^k\}) = \sum_{j=1}^{m} \pi_{jk} P_F(\{s^j\}) \quad \text{for all } k. \tag{12}$$

With respect to $\mathscr{U}_{inc}$, F dominates G if and only if there are numbers π_{jk} with (10), (11), (12), and, in addition,

$$\sum_{j \in B_k} \pi_{jk} = 1 \quad \text{for all } k, \tag{13}$$

where $B_k = \{j | s^k \le s^j\}$, and $\le$ is the usual ordering in $\mathbb{R}^n$. Hence, strong FSD holds if and only if a linear program with restrictions (10) to (13) has a feasible solution.

Similarly, with respect to $\mathscr{U}_{iconc}$, F dominates G if and only if there are numbers π_{jk} with (10), (11), (12), and

$$\sum_{k=1}^{m} \pi_{jk}\, s^k \le s^j \quad \text{for all } j. \tag{14}$$

Again, this is equivalent to the existence of a feasible solution to a linear program.

Next, with respect to $\mathscr{U}_{conc}$ the same applies with (14) replaced by

$$\sum_{k=1}^{m} \pi_{jk}\, s^k = s^j \quad \text{for all } j. \tag{15}$$

The remaining strong SSD relations are put down to the above relations as follows. F dominates G with respect to $\mathscr{U}_{conv}$ if and only if G dominates F with respect to $\mathscr{U}_{conc}$. F dominates G with respect to $\mathscr{U}_{iconv}$ if and only if $\bar{G}$ dominates $\bar{F}$ with respect to $\mathscr{U}_{iconc}$. Here, we notate $\bar{F}(x) = P_F(\{z \in \mathbb{R}^n | z \ge -x\})$, and $\bar{G}$ analogously.

Weak FSD and SSD are connected with inequalities on the distribution functions and their n-dimensional integrals. See the reviews in Mosler [13] and Mosler and Scarsini [16]. F dominates G with respect to $\mathscr{G}_1^-$ ($\mathscr{G}_1^+$) if and only if $F(z) \le G(z)$ ($\bar{F}(z) \ge \bar{G}(z)$) for all $z \in \mathbb{R}^n$. Dominance with respect to $\mathscr{F}_2^-$ and, equivalently, to $\mathscr{G}_2^-$ is true if and only if

$$\int_{-\infty}^{z} F(t)\mathrm{d}t \le \int_{-\infty}^{z} G(t)\mathrm{d}t$$

for all $z \in \mathbb{R}^n$. Similarly, with respect to $\mathscr{F}_2^+$ and, equivalently, to $\mathscr{G}_2^+$ dominance is true if and only if

$$\int_z^\infty \bar{F}(t)\mathrm{d}t \geq \int_z^\infty \bar{G}(t)\mathrm{d}t$$

for all $z \in \mathbb{R}^n$. With finitely discrete distributions, each of these conditions can be checked by passing through the grid of points in $\mathbb{R}^n$ which is generated by the joint support S. This is done here with a new efficient algorithm, see Dyckerhoff et al. [5].

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