Introduction

A bibliography on stochastic orderings. Was there a real need for it? In a time of reference databases as the MathSci or the Science Citation Index or the Social Science Citation Index the answer seems to be negative. The reason we think that this bibliography might be of some use stems from the frustration that we, as workers in the field, have often experienced by finding similar results being discovered and proved over and over in different journals of different disciplines with different levels of mathematical sophistication and accuracy and most of the times without cross references. Of course it would be very unfair to blame an economist, say, for not knowing a result in mathematical physics, or vice versa, especially when the problems and the languages are so far apart that it is often difficult to recognize the analogies even after further scrutiny. We hope that collecting the references on this topic, regardless of the area of application, will be of some help, at least to pinpoint the problem.

We use the term stochastic ordering in a broad sense to denote any ordering relation on a space of probability measures. Questions that can be related to the idea of stochastic orderings are as old as probability itself. Think for instance of the problem of comparing two gambles in order to decide which one is more favorable.

The first body of work in which a concept of stochastic ordering is central is probably the one due to Hardy, Littlewood and Pólya in the twenties and thirties which culminated with the first edition of their book on inequalities (1934). Their idea of majorization is not formulated as a stochastic ordering, but the translation is immediate by associating to a vector in $\mathbb{R}^n$ the discrete probability measure that puts mass $1/n$ on each of the components of the vector.

Majorization is a way of comparing two nonnegative vectors (of the same dimension) in terms of the dispersion of their components: Given $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\mathbf{x} \leq_{\mathbf{maj}} \mathbf{y}$ iff

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \qquad \forall k = 1, \ldots, n, \qquad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)},$$

where $(x_{(1)}, \ldots, x_{(n)})$ is the decreasing rearrangement of the vector $\mathbf{x}$.

Hardy, Littlewood and Pólya provided interesting characterizations of majorization, drawing from some ideas due to Muirhead (1903) and Schur (1923). For instance, the following conditions are equivalent

i. $\mathbf{x} \leq_{\mathbf{maj}} \mathbf{y}$.

ii. $\mathbf{x} = \Pi \mathbf{y}$, for some doubly stochastic matrix Π.

iii. $\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$ for all g convex.

Another important paper in the history of stochastic orderings is due to Karamata (1932), who provided an analysis of concepts that will then be known as dilation and second degree stochastic dominance, both of which can be considered as a generalization of majorization.

After World War II more results have been discovered and new areas of research have been developed. Many results now originate from applications: for instance Blackwell's

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analysis is motivated by the statistical problem of comparing experiments (see Blackwell 1951, 1953). This lead him to the formulation of some results in the theory of dilation, which several people in the following decade have generalized, among them Choquet(1962), Cartier et al. (1964), Strassen (1965), etcetera. This theory extends the theory of majorization to general probability measures. In this respect it can be seen as the first authentic development of a theory of stochastic orderings per se (namely an ordering relation on a space of probability measures). Very briefly, the theory of dilation proves the equivalence of the following statements concerning two probability measures P_1, P_2 on a measurable linear space $(\mathcal{X}, \mathcal{A})$:

1. $\int \phi \, dP_1 \leq \int \phi \, dP_2 \quad \forall \phi$ convex.
2. There exists a Markov kernel $M(\cdot, \cdot)$ such that $\int x \; M(y, dx) = y$ and
$$P_1(A) = \int M(y, A) \; P_2(dy) \qquad \forall A \in \mathcal{A}.$$
3. There exist two random variables X_1, X_2 on a probability space $(\Omega, \mathcal{F}, P)$, with values in $(\mathcal{X}, \mathcal{A})$ such that $\mathcal{L}(X_1) = P_1$, $\mathcal{L}(X_2) = P_2$ and $\mathrm{E}(X_2|X_1) = X_1$ P-a.s..

It is interesting to notice that the properly said stochastic order (namely the simplest and most intuitive form of stochastic ordering) was studied only later. Lehmann (1955) introduced the order for distributions on $\mathbb{R}^n$ in connection with some inferential problems. The idea has been subsequently generalized and several characterizations have been proved. The most general and thorough analysis of the stochastic order can be found in Kamae, Krengel and O'Brien (1977) whose scheme we follow here. Given two probability measures P_1, P_2 on a partially ordered Polish space $(\mathcal{X}, \mathcal{A})$, $P_1 \leq_{\text{st}} P_2$ iff any of the following equivalent conditions holds.

i. $\int \phi \, dP_1 \leq \int \phi \, dP_2 \qquad \forall \phi$ increasing.

ii. $P_1(A) \leq P_2(A)$ for all upper sets A.

iii. There exists a Markov kernel $M(\cdot, \cdot)$ such that $\int M(y, dx) \geq y$ and
$$P_2(A) = \int M(y, A) \; P_1(dy) \qquad \forall A \in \mathcal{A}.$$

iv. There exist two random variables X_1, X_2 on a probability space $(\Omega, \mathcal{F}, P)$ with values in $(\mathcal{X}, \mathcal{A})$ such that $\mathcal{L}(X_1) = P_1$, $\mathcal{L}(X_2) = P_2$ and $X_1 \leq X_2$ P-a.s..

The generality of the setting allows the above definitions to be used for comparing (the law of) real valued random variables, random vectors, stochastic process, etc..

The stochastic order, the dilation order, and their variations are the orderings that have generated the most work both in theoretical and applied sense. Several other orderings have been defined and studied, motivated by different areas of applications.

Queueing theory has provided many applications for the comparison of stochastic processes. See Stoyan (1977, 1983). In reliability theory, the need often arises to compare the lifelengths of some complex systems (or of some components of the systems). The original studies (e.g., Barlow, Marshall and Proschan 1963) were rarely related to stochastic orderings in a proper sense, since they did not compare two generic elements of a set of probability measure, but rather one generic measure and another fixed reference measure. For instance the concepts of new better than used or increasing failure rate implicitly set a comparison with the exponential distribution. It is often useful to compare the lifelength of a system whose components are not necessarily independent with the lifelength of a system whose components have the same marginal distributions as the components of the original system, but are independent. The concept of association (Esary, Proschan and Walkup 1967) originates by similar ideas. A random vector $(X_1, \ldots, X_n)$ is associated iff $\mathrm{Cov}(h(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0$ for all increasing functions $h, g : \mathbb{R}^n \to \mathbb{R}$. Several other concepts of dependence have been proposed and contrasted, all of them based on the comparison between a multivariate distribution and the corresponding distribution with independent marginals (see e.g. Lehmann 1966). It is not until later, though, that these concepts have been given the form of a stochastic ordering relation. See e.g. Schriever (1985) for association, and Kimeldorf and Sampson (1987) for other concepts of dependence.

An idea that is strictly related to association, but was proposed independently with motivations deriving from mathematical physics is the so called FKG inequality (Fortuin, Kastelyn and Ginibre 1971). Again it is only later that the two concepts have been related to each other.

After the stochastic order, other ordering relations have been introduced to satisfy statistical necessities. Among these we mention the dispersion and spread orderings of Bickel and Lehmann (1976, 1979) and others in the following years.

On the mathematical side it is worth mentioning a considerable amount of work about the comparison of solutions of stochastic differential equations. Given Kamae, Krengel and O'Brien's almost sure characterization of stochastic order, it is possible to re-interpret all these results as stochastic comparison of random processes defined as solutions of stochastic differential equations.

Last came the economists. A bulk of contributions in this area originated from the two articles by Hadar and Russell (1969) and Hanoch and Levy (1969). However, similar ideas date back to Massé and Morlat (1953), Quirk and Saposnik (1962), and Fishburn (1964). The problems that these authors deal with can be formulated as follows: several expected utility maximizers have to choose between two prospects. Under which conditions on the distribution functions of two prospects will all of the agents make the same decision? Or alternatively, under which conditions will a decision maker (whose utility function is only partially specified) prefer one prospect over the other? Answers were first provided for the classes of increasing and increasing concave utility functions. These results are known under the name of first and second degree stochastic dominance. The above characterizations were then extended to other classes of utility functions representing different kinds of risk postures.

Immediately after these articles on stochastic dominance, the papers by Rothschild and Stiglitz (1970, 1971) and Diamond and Stiglitz (1974) dealt with the problem of comparing risks.

The relevance of the early contributions to stochastic dominance and the comparison of risks in general does not lie in their mathematical novelty: The results were often either sloppily stated or well known under more general conditions. But it is undeniable that they gave rise to a huge amount of literature in various areas of economic theory.

Some of the most interesting articles concern the optimal composition of a portfolio under different assumptions on the distribution of the assets and the attitude towards risk of the investors. Another interesting area of applications is the measurement of inequality. Kolm (1969) and Atkinson (1970) gave a coherent mathematical form to some basic principles for inequality measurement and its characterizations. It is worth remembering that the first contributions concerning the comparison of income distributions date back to Lorenz (1905), whose concentration curve is basic to define what is now known as Lorenz ordering, and to Dalton (1920) who introduced the principle of transfer.

What is to be found of all the above in this bibliography? We deeply hope most of it. We are well aware that many papers might have slipped our attention, especially if the involvement of stochastic orderings was minor or if they were published in difficult-to-find journals. But we tried to include everything that was about the theory and/or some applications of stochastic orderings, or that merely used some relevant fact about stochastic orderings in a substantial way, and was published before 1992 included. Of course, the last year is not fully covered as some 1992 publications are still in print. Several fields, like stochastic processes, reliability, queues, appear only as far as stochastic orderings are involved. E.g., not all comparison results of stochastic processes are included. Some other fields, like unimodality, are fully covered.

What is not to be found here? When deciding what to include in the bibliography, we had to draw a line, and where to draw it was not an easy decision. Some general criteria were used throughout. Papers dealing with probability inequalities per se (e.g. Chebyshev, Markov, Hölder, etc.), without reference to an ordering, were not included. Inference under order restrictions is only partially covered. Also, in economics, the vast literature on inequality indices has been omitted; the reader is referred to Giorgi's (1992a,b) comprehensive bibliography on the Gini index. Papers about the economics of risk and risk aversion were not included, except for some fundamental ones and for the articles that use the theory of stochastic orderings, for instance the ones that link the comparison of risk aversion and the comparison of risk. Papers in the economics of information have been omitted, too. The excellent book by Nermuth (1982) will help the reader into the field and the relevant literature. We included a few applications to physics and to the life sciences, but we believe that there exist many more which we are not aware of.

The Appendix contains some grey literature, contributions to non-refereed publications and recent technical reports which we received from colleagues. Some papers and books published after 1992 appear in the Appendix, too, but there has been absolutely no attempt to be exhaustive from 1993 on.

No cleaning has been performed on the material that qualified for inclusion (namely published articles on stochastic orderings). Therefore a title appears in the list even if the paper contains errors or duplications of known results.

This is not the first bibliography on stochastic orderings. Bawa (1982) already contained more than 400 titles. Other interesting sets of references are contained in the books by Tong (1980) on probability inequalities, Marshall and Olkin (1979) on majorization, Stoyan (1977, 1983) on stochastic systems. These books are also excellent sources of results. Levy (1992) contains a bibliography of applications to finance. The reader can find other interesting papers (of technical or survey nature) in the collections edited by Whitmore and Findlay (1978), Tong (1984), Block, Sampson and Savits (1990), Mosler and Scarsini (1991), Shaked and Tong (1993). A concise first introduction to stochastic orderings can be found in Ross (1983, Chap. 8).

The book is organized as follows. Each paper has been classified according to a list of fields (the same paper may appear in more than one field). The main list contains all the papers in alphabetic order (by the first author), with the complete references and the proper fields. Usually, key words are given only when provided in the articles. A second list includes the papers classified by fields. The Appendix contains the post-1992 papers and the grey literature.

We thank all colleagues who have contributed to this bibliography. Several people helped by sending their reprints. We do not mention their names here for fear of forgetting some. Of course we did not include all the titles that have been submitted to our attention. We cannot name the many students who typed the material into the computer. Rainer Dyckerhoff and Hartmut Holz supervised this process, wrote many nice computer programs, and produced the final layout. Finally, we thank the editors of the Lecture Notes in Economics and Mathematical Systems and Dr. Müller of Springer Verlag for including the bibliography in this series.

For the future development of the bibliography we would greatly appreciate receiving any new material. Also, corrections of mistakes and omissions are welcome.

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