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Multiattribute utility functions, partial information on coefficients, and efficient choice

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Summary. The expected utility analysis of decision under risk needs information on the alternatives and on the decision maker's preferences which in many practical situations are difficult to obtain. This paper presents a procedure for choosing between multiattribute risky alternatives when the probabilities of outcomes are known, the utility function is general multilinear (i.e., can be decomposed into sums and products of univariate utility functions), and there is some partial information on univariate utilities (viz. increasingness) and arbitrary partial information on the scaling coefficients. Pairwise comparisons in the set of alternatives yield a subset which is efficient under the given partial information. Additive and multiplicative utility functions are special cases of the multilinear one. The paper gives particular attention to linear partial information (LPI) on coefficients, which is obtained by standard assessment procedures. The approach can be combined with dominance procedures which use other partial information as LPI on probabilities.

Zusammenfassung. Betrachtet werden Risikoentscheidungen bei mehreren Attributen. Für die Bestimmung des erwarteten Nutzens der Alternativen benötigt man Informationen über die Präferenzen des Entscheidungsträgers, die in konkreten Anwendungen häufig nur schwer zu beschaffen sind. Im folgenden Artikel wird ein Verfahren vorgestellt, mit dem man bereits Entscheidungen treffen kann, wenn die Risikonutzenfunktion allgemein multilinear ist (d. h. in Summen und Produkte von univariaten Nutzenfunktionen dekomponiert werden kann) und eine bestimmte unvollständige Information über die univariaten Nutzenfunktionen (nämlich monotones Wachstum) und beliebige unvollständige Information über die Skalenfaktoren vorliegt. Aus Paarvergleichen in der Menge der Alternativen erhält man eine bezüglich der gegebenen Information effiziente Teilmenge. – Additiv bzw. multiplikativ dekomponierte Nutzenfunktionen ergeben sich als Spezialfälle der multilinearen Form. Der Artikel behandelt eingehend die lineare partielle Information (LPI) über die Skalenfaktoren, die sich aus den üblichen Verfahren zur praktischen Ermittlung der Nutzenfunktion ergibt. Der Ansatz kann mit Dominanzverfahren kombiniert werden, die auf andere Arten unvollständiger Information (etwa auf LPI über die Wahrscheinlichkeiten) zurückgreifen.

Key words: Decision theory, multiattribute choice, stochastic dominance, multilinear utility decomposition, linear partial information

Schlüsselwörter: Entscheidungstheorie, mehrere Attribute, stochastische Dominanz, multilineare Nutzendekomposition, lineare partielle Information

1. Introduction

The expected utility analysis of a decision under risk needs information on the alternatives and on the decision maker's preferences which in many practical situations is difficult to obtain. Therefore, many decision procedures have been proposed which are based on partial information only. In particular, every interactive decision procedure employs partial information in a sequential way. The purpose of this paper is to present a new procedure for choosing between multiattribute alternatives when the probabilities of outcomes are known. It is assumed that the utility function can be decomposed into sums and products of univariate utility functions and that there is some partial information on the univariate utilities (increasingness) and arbitrary partial information on the scaling coefficients.

In multiattribute utility theory (MAUT), choice between alternatives $X=(X_1, X_2, \dots X_k)$ is analyzed where the X_i are real-valued random variables. Here, X is called a *(risky) alternative*, or a *prospect*, a realization $x=(x_1, x_2, \dots x_k)$ of X is called an *outcome* or a *consequence*, every $i \in \{1, 2 \dots k\}$ is called an *attribute*. In the sequel we speak of alternatives, outcomes, and attributes. A decision maker (DM) is supposed to choose between alternatives according to expected utility $\mathrm{Eu}(X_1, X_2, \ldots X_k)$ with some k-variate utility function. MAUT decision methodology proceeds as follows: First, some utility independence properties of the DM's weak preference on alternatives are assessed. They yield a particular decomposition of the k-variate utility function u into sums and products of utility functions which have less arguments. Second, these utility functions plus a number of scaling constants are fully assessed by asking questions to the DM. Finally, $\mathrm{Eu}(X_1, X_2, \ldots X_k)$ is evaluated for every possible alternative.

In the sequel we assume that the DM's k-variate utility function $u(x_1, \ldots x_k)$ can be decomposed into sums and products of k univariate utility functions $u_1(x_1), \ldots u_k(x_k)$ and real coefficients. With other words, u has a general *multilinear decomposition.* E.g., when $k=3, u(x_1, x_2, x_3) = \alpha_1 u_1(x_1) + \alpha_2 u_2(x_2) + \alpha_3 u_3(x_3) + \alpha_{12} u_1(x_1) u_2(x_2) + \alpha_{23} u_2(x_2) u_3(x_3) + \alpha_{13} u_1(x_1) u_3(x_3) + \alpha_{123} u_1(x_1) u_2(x_2) u_3(x_3)$. A sufficient condition for this is that each attribute is utility independent of the set of remaining attributes; see Keeney and Raiffa (1976, p. 293), Fishburn (1973), and Farquhar (1975). The *additive decomposition* $u(x_1, \ldots x_k) = \Sigma\, \alpha_i u_i(x_i)$ and the *multiplicative decomposition* $u(x_1, \ldots x_k) = \Pi\, \alpha_i u_i(x_i)$ are special cases.

In most practical applications reported in the literature, additive (and sometimes multiplicative) decompositions have been used. But the additive model cannot mirror preferential dependencies between attributes, and the multiplicative model is similarly inadequate as it shows two special kinds of dependency only, viz. bivariate risk aversion and proneness (Richard 1975). The multilinear model is much better adapted to practical situations. However, to specify the k-variate multilinear utility function of a given DM, k univariate utility functions and up to $2^k - 1$ coefficients have to be assessed, which can be a tedious if not impossible task. Instead, we propose to use partial information on coefficients and incompletely specified univariate utility functions. An alternative which proves to be inferior to another one with respect to every utility function of this incomplete specification may be removed from the set of alternatives. Pairwise comparisons in the set of alternatives yield a subset which is efficient with respect to the given partial information. Then, it may be decided whether to assess more information on the utility function or to choose by some other device from the efficient set. The additional information may be complete (all coefficients and univariate utility functions are known) which results in a simple expected utility maximization, or, the information may be partial again which yields an efficient set which is not larger and possibly smaller than the previous one.

In this paper, by *information* or *partial information* we simply mean a set of utility functions. Given some partial information, we say that alternative Y *dominates* alternative X *with respect to the information* if and only if the expected value of utility differences is non-negative, $E[u(Y) - u(X)] \geq 0$, for all utility functions u in that set, as far as the expected value exists.

The concept of partial information has been introduced to expected utility in a variety of ways. Fishburn (1964, 1965) addresses the problem of comparing alternatives when there is full information on the utility function but only partial information on probabilities of possible outcomes, viz. inequalities between probabilities and sums of probabilities. He presents a method (called "method of equating coefficients") to derive dominance conditions in terms of utility scores under this information. Fishburn's approach has been revisited and reinvented by several succeeding authors, among them Hannon (1981), Bromage (1982) and others. Linear partial information (LPI) on probabilities with full information on utilities has also been investigated by Kofler and Menges (1976), Ehemann (1981), and Kofler et al. (1984) who maximize the minimum expected utility (minimum with respect to the given information). Weber (1987) gives a survey of decision making with partial information.

With multiple attributes, a general approach to MAUT via efficient sets has been proposed in several papers by White and Sage (White and Sage 1980; Sage and White 1984). They employ graphtheoretic methods and indicate that the approach may be used for LPI on both probabilities and coefficients of an additive utility function. Jacquet-Lagrèze and Siskos (1982) assume an additive decomposition and present an approach (based on linear regression) to determine not only one utility function but a set of functions which is consistent with the assessed information. Korhonen et al. (1984) and Köksalan (1989) use cones of inferior alternatives in an interactive way. Kirkwood and Sarin (1985) and Hazen (1986) present decision procedures which rely on known univariate utilities and LPI about utility coefficients. While Kirkwood and Sarin (1985) do not go beyond additive utility decompositions and some special cases of LPI, Hazen (1986) also includes multiplicative utilities and general LPI.

Dominance of single-attribute alternatives with respect to all increasing utility functions has been investigated in an economic context by many authors starting from Hanoch and Levy (1969) and Hadar and Russell (1969); see also Vickson (1977) who employs some kind of linear information on utilities. For recent references in a multiattribute setting, see Mosler (1984) and Scarsini (1988). In connection with LPI on probabilities (but not on utility coefficients), related dominance results have been presented by Pearman and Kmietowicz (1986) and Keppe and Weber (1990).

In this paper partial information about general multilinear utility functions is investigated. Full information on probabilities is assumed. No information on univariate utilities is used besides increasingness and standardization with respect to two reference outcomes, while partial information on coefficients is arbitrary. (Closed form results, however, are mainly obtained for LPI concerning the coefficients.) The assumption of increasing univariate utilities is motivated by the fact that in almost all reasonable circumstances it will be possible to rank-order the levels of every single attribute while in many cases assessing univariate cardinal utilities may be difficult or costly. Section 2 introduces the general multilinear utility decomposition and the main cases of partial information that can be obtained by the usual assessment procedures (comparisons of lotteries). Section 3 deals with standard additive decompositions and exhibits sufficient and necessary conditions for dominance of alternative Y over alternative X. They are formulated in terms of certain probabilities of Y and X. Section 4 attacks the general case of dominance conditions for multilinear utilities (including multiplicative ones), and Sec. 5 concludes the paper.

2. Multilinear utility functions and the assessment of partial information

The general multilinear decomposition is given by

$$u(x_1, x_2, \dots x_k) = \sum_{\substack{I \subset K \\ I \neq \emptyset}} \alpha_I \prod_{i \in I} u_i(x_i), \tag{2.1}$$

where the sum extends over all subsets of attributes $I \subset K = \{1, 2, \dots k\}$ which are not empty. α_I denotes a multiindexed constant, $\alpha_I = \alpha_{i_1 \dots i_m}$ when $I = \{i_1, \dots i_m\}$,the u_i are increasing functions, $u_i: C_i \to \mathbb{R}$, and the C_i are intervals. If there exist two fixed outcomes $a = (a_1, \dots a_k)$ and $b = (b_1, \dots b_k)$ with

$$u_i(a_i) = 0, \quad u_i(b_i) = 1, \quad i = 1, \dots k \tag{2.2}$$

$$u(a) = 0, \quad u(b) = 1 \tag{2.3}$$

Eq. (2.1) is called a *standard multilinear decomposition* of u.

Two special cases are important: the *standard additive decomposition*

$$u(x_1, x_2, \dots x_k) = \sum_{i=1}^{k} \alpha_i u_i(x_i), \tag{2.4}$$

where $\alpha_I = 0$ has been chosen in (2.1) for $|I| > 1$ (i.e., when I has more than one element) and the *standard multiplicative decomposition*

$$u(x_1, x_2, \dots x_k) = \sum_{i \in K} \alpha_i u_i(x_i) + \beta \sum_{\substack{I \subset K \\ |I| \geq 2}} \beta^{|I|-2} \prod_{i \in I} \alpha_i u_i(x_i), \tag{2.5}$$

where $\alpha_I = \beta^{|I|-1} \prod_{i \in I} \alpha_i$ for $|I| > 1$ has been chosen in (2.1). Observe that (2.5) specializes to (2.4) when $\beta = 0$.

The class of additive utility functions is limited to situations where the DM perceives no interaction between the attributes. The multiplicative class is closely related to that since each strictly multiplicative decomposition (2.5) with $\beta \neq 0$ can be written as a product $u(x_1, \dots x_n) = \Pi\, w_i(x_i)$ with some w_i, hence the logarithmic utility has an additive decomposition. The general multilinear form (2.1) is much better suited to mirror a given preferential dependency between the attributes. To yield a multilinear decomposition (2.1) we have to verify that in the DM's preference relation every single attribute i is utility independent from the set of remaining attributes $K \setminus \{i\}$. To yield a multiplicative decomposition (2.5) we need more, viz. that at least every pair $\{i, j\}$ of attributes is utility independent from $K \setminus \{i, j\}$; see Keeney and Raiffa (1976, pp. 289), Fishburn (1973), Farquhar (1975). E.g., the utility function

$$u(x_1, x_2, x_3) = 5/6\, x_1 - 1/6\, x_2 x_3 + 1/3\, x_1 x_2 x_3, \quad x_i \in [0, 1]$$

represents a preference relation which cannot be represented by a multiplicatively decomposed utility function. (Note that $\{2, 3\}$ is not utility independent from $\{1\}$.)

However, to specify a general multilinear utility function completely, k univariate utility functions and $2^k - 1$ constants have to be determined. This is normally done by offering special pairs of alternatives to the DM and asking her to make hypothetical choices. By

$$F_X: \begin{bmatrix} p_1 & \dots & p_n \\ (x_1^{(1)}, \dots x_k^{(1)}) & \dots & (x_1^{(n)}, \dots x_k^{(n)}) \end{bmatrix}$$

we denote the alternative which yields outcome $(x_1^{(i)}, \dots x_k^{(i)})$ with probability p_i, $i = 1, \dots n$, and by (b_J, a_{-J}) we denote the outcome with levels b_i in all attributes $i \in J$, a_i in all attributes $i \in -J$ where $-J = K \setminus J$. From (2.1), (2.2), and (2.3) follows that

$$u(b_J, a_{-J}) = \sum_{I \subset J} \alpha_I, \; J \subset K, \; J \neq \emptyset, \tag{2.6}$$

and, in particular, with $J = \{j\}$

$$u(b_j, a_{-j}) = \alpha_j, \quad j \in K. \tag{2.7}$$

Therefore, if the DM prefers (b_J, a_{-J}) for sure at least as (b_L, a_{-L}) for sure we get

$$\sum_{I \subset J} \alpha_I \geq \sum_{I \subset L} \alpha_I. \tag{2.8}$$

and, in particular, with $J = \{j\}$, $L = \{l\}$

$$\alpha_j \geq \alpha_l. \tag{2.9}$$

If she is indifferent between the alternatives

$$F_{Y'}: \begin{bmatrix} 1 \\ (b_J, a_{-J}) \end{bmatrix} \quad \text{and} \quad F_{X'}: \begin{bmatrix} 1-\delta & \delta \\ a & b \end{bmatrix}$$

for some δ, $0 \leq \delta \leq 1$, we conclude

$$\sum_{I \subset J} \alpha_I = \delta. \tag{2.10}$$

Also, if the DM is indifferent between

$$F_{\hat{Y}}: \begin{bmatrix} 1 \\ (c_j, x^*_{-j}) \end{bmatrix} \quad \text{and} \quad F_{\hat{X}}: \begin{bmatrix} 1-\delta & \delta \\ (a_j, x^*_{-j}) & (b_j, x^*_{-j}) \end{bmatrix}$$

(where c_j, a_j, and b_j refer to the j-th attribute and x^*_{-j} to the rest) for some c_j, x^*_{-j}, and δ, $0 \leq \delta \leq 1$, we have the same indifference for a_{-j} in place of x^*_{-j}. Therefore,

$u(c_j, a_{-j}) = a_j\delta.$

Since $u(c_j, a_{-j}) = \alpha_j u_j(c_j)$, we get

$$u_j(c_j) = \delta \tag{2.11}$$

if $\alpha_j > 0$. Similarly, we may have assessed $u(c_k, a_{-k}) = \alpha_k \varepsilon$ for some ε. If the DM prefers (c_j, a_{-j}) for sure at least as (c_k, a_{-k}) for sure we get the inequality

$$\delta \alpha_j \geq \varepsilon \alpha_k \tag{2.12}$$

(with known coefficients δ and ε).

It should be noted that information of inequality type, (2.8), (2.9), and (2.12) is much more easily assessed than that of equality type (2.10) and (2.11).

In principle, from equalities like (2.10) and (2.11) the univariate utilities and the constants can be determined up to any desired accuracy; see Keeney and Raiffa (1976, pp. 277). In practice, this approach can be tedious if not impossible due to limitations of the DM's response. The DM may (1) not have the time to answer all the hypothetical tradeoff questions needed for a complete assessment. Also, she may (2) not be willing or (3) not be able to respond to certain questions. (4) if there are several decision makers, they may agree on some questions but disagree on others. Instead, we propose to collect and use information of type (2.8) to (2.12) in a sequential way. At every step of the procedure the information which has been collected so far defines a class of utility functions u. An alternative which proves to be inferior to another one with respect to every utility function in this class may be removed from further consideration. Pairwise comparisons in the set of alternatives yield a subset which is efficient with respect to the given information. Then, it may be decided whether to assess more incomplete information (and use it in a next step) or to assess complete information or to choose by some other device from the efficient set.

3. Partial information on an additive utility

In this section, partial information on the DM's utility function u is assumed as follows: First, u has a standard additive decomposition, i.e., (2.2) to (2.4) hold for some given fixed outcomes a and b. Second, all univariate utilities are increasing. Third, the unknown coefficients α_i are positive. Fourth, the vector of coefficients $(\alpha_1, \alpha_2, \dots \alpha_k)$ belongs to some given set of coefficient vectors. Dominance criteria are presented with respect to this information. Specific coefficient sets of practical interest are investigated in detail.

Theorem 1 (Additive case): *Assume that the utility function has a standard additive decomposition with $u_i(a_i) = 0$, $u_i(b_i) = 1$ for two fixed outcomes $(a_1, \dots a_k)$ and $(b_1, \dots b_k)$, further that univariate utilities are increasing and the coefficient vector $(\alpha_1, \alpha_2, \dots \alpha_k)$ belongs to some given nonempty set A, $\alpha_i > 0$ for all i. With respect to this information an alternative Y dominates an alternative X if and only if for all z_i, i $= 1, 2, \dots k$,*

$$\inf_{\alpha \in A} \sum_{i=1}^{k} \alpha_i \delta_i(z_i) \geq 0 \tag{3.1}$$

holds, where $\delta_i(z_i)$ is defined by

$$\delta_i(z_i) = P(Y_i \geq z_i) - P(X_i \geq z_i). \tag{3.2}$$

Theorem 1 is not proved here since it will appear as a special case of Theorem 2 below. The assumption of *a priori* positive coefficients α_i seems natural in applications; however, it can be easily dropped (cf. Theorem 2). Observe that δ_i denotes the difference of two marginal cumulative distribution functions. So, our dominance criterion (3.1) states that the weighted sum of these differences, weighted by any admissible coefficients, must be nonnegative everywhere.

With vectors $\alpha = (\alpha_1, \dots \alpha_k)$ and $\delta(z) = (\delta_1(z_1), \dots \delta_k(z_k))$, α' denoting the transpose of α, (3.1) can be written as $\delta(z)\alpha' \geq 0$ for all $\alpha \in A$ or, equivalently,

$$\delta(z) \in A^*, \tag{3.3}$$

where A^* is the polar cone of A, $A^* = \{\beta \in \mathbb{R}^k | \beta\alpha' \geq 0$ for all $\alpha \in A\}$.

In order to compare two given alternatives X and Y when A is some set of coefficient vectors, we have to compute $\delta(z)$ and to check optimal values of the program in (3.1) for every $z = (z_1, \dots z_k)$. (If X and Y have finitely discrete distributions then $\delta(z)$ assumes a finite number of values, and the program (3.1) must be solved at finitely many z only.) Alternatively, we may compute A^* and check (3.3) for every z. In the case when A is defined through linear inequalities[1],

$$A = \{\alpha | \alpha W \geq v\} \tag{3.4}$$

with some $k \times r$ matrix W and right hand side vector v, the programs are linear programs. If A has form (3.4), we say that A is a *linear partial information* (LPI) on coefficients. If in addition $v = 0$, A is called a *conical LPI*. In particular, the assessment of (in)equalities of types (2.6) to (2.12) from actual preference statements of the DM produces some linear information A. If (2.10) is not used, the information is conical linear. This is a most relevant case.

In the conical case, the polar cone is $A^* = \{\gamma W' | \gamma \geq 0\}$; see, e.g., Bazaraa and Shetty (1979). The following lemma on conical linear information yields dominance criteria for a number of practically important cases.

Lemma 1. *Assume $A \subset \{\alpha | \alpha W \geq 0\} \neq \emptyset$ with some $k \times r$ matrix W having rank k. Let $\tilde{W} = (WW')^{-1} W$. Then, for any z*

[1] $\geq$ ($>$) between vectors means $\geq$ ($>$) between all components.

(*i*)

$$\delta(z)\tilde{W} \geq 0 \tag{3.5}$$

is sufficient for (3.1) to hold.
(*ii*) *If, in addition,* $A = \{\alpha \mid \alpha W \geq 0\}$ *and* $W'\tilde{W} \geq 0$,[2] *(3.5) is necessary and sufficient.*

Obviously, if W is a square matrix and of full rank, we have $k = r$, $\tilde{W} = (W^{-1})'$, and $W'\tilde{W} = I_r \geq 0$.

Proof. Note that $\tilde{W}W' = I_k$. For every z we have $\delta(z)\alpha' = \delta(z)\tilde{W}W'\alpha' = \delta(z)\tilde{W}(\alpha W)'$. (i) For every $\alpha \in A$ $\alpha W \geq 0$ holds. So, from (3.5) we conclude $\delta(z)\alpha' \geq 0$ for every $\alpha \in A$; hence (3.1). (ii) Let $A = \{\alpha \mid \alpha W \geq 0\}$ and $W\tilde{W} \geq 0$. If (3.1) holds, we necessarily get $\delta(z) \in A^* = \{\gamma W' \mid \gamma \geq 0\}$, i.e., $\delta(z) = \gamma W'$ with some $\gamma \geq 0$. Therefore $\delta(z)\tilde{W} = \gamma W'\tilde{W} \geq 0$, hence (3.5). QED

Corollary 1. *Let the utility function be as in Theorem 1. When A is defined through inequalities as follows, then necessary and sufficient for dominance of an alternative Y over an alternative X is that for all* z_i

(*i*) $\delta_i(z_i) \geq 0$, $i = 1, \dots k$, *holds, provided*
$A = \{\alpha \mid \alpha_i \geq 0, i = 1, \dots k\}$,

(*ii*) $\sum_{i=1}^{j} \delta_i(z_i) \geq 0, j = 1, \dots k$, *holds, provided*
$A = \{\alpha \mid \alpha_1 \geq \alpha_2 \geq \dots \alpha_k \geq 0\}$,

(*iii*) $\sum_{i=1}^{j} \delta_i(z_i) \geq 0, j = 1, \dots l$, *and* $\sum_{i=l+1}^{m} \delta_i(z_i) \geq 0$,
$m = l + 1, \dots k$ *holds, provided*
$A = \{\alpha \mid \alpha_1 \geq \alpha_2 \geq \dots \alpha_l \geq 0, \alpha_{l+1} \geq \alpha_{l+2} \geq \dots \alpha_k \geq 0\}$

(*iv*) $\sum_{i=1}^{k} \delta_i(z_i) \geq 0$ *and* $\delta_i(z_i) \geq 0$, $i = 1, \dots k-1$, *holds,*
provided $A = \{\alpha \mid \alpha_i \geq \alpha_k \geq 0, i = 1, \dots k-1\}$,

(*v*) $\sum_{i=1}^{j} \frac{1}{w_i} \delta_i(z_i) \geq 0, j = 1, \dots k$, *holds, provided*
$A = \{\alpha \mid w_1\alpha_1 \geq w_2\alpha_2 \geq \dots w_k\alpha_k \geq 0\}$ *where* $w_1, w_2, \dots w_k$ *are fixed positive weights.*

Proof. Each a in the corollary denotes a conical linear information on coefficients $A = \{\alpha \mid \alpha W \geq 0\}$. In (i) we have $W = I$; putting $\tilde{W} = I$ in the lemma yields (3.1) if and only if $\delta_i(z_i) \geq 0$, for all $i = 1, \dots k$. Theorem 1 tells that Y dominates X with respect to the information $\mathring{A} = \{\alpha \in A \mid \alpha_i > 0, i = 1, \dots k\}$ if and only if for all z_i (3.1) holds with $\mathring{A}$ instead of A. But, as $\mathring{A}$ is dense in A, dominance with respect to $\mathring{A}$ is equivalent to dominance with respect to A, and (3.1) with $\mathring{A}$ is equivalent to (3.1) with A. Parts (ii) to (v) of Corollary 1 are proved similarly. Note that in all these cases we have $r = k$ and $\tilde{W} = (W^{-1})'$; e.g., in (iv)

$$W = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 1 \end{pmatrix}, \quad \tilde{W} = \begin{pmatrix} 1 & 0 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

QED.

In part (i) of Corollary 1, A represents the case of null information on coefficients. In (ii) to (iv), A is determined by assessing inequalities of type (2.9); and in (v) by assessing those of types (2.9) or (2.12). Of course, the cases treated are examples only. In particular, we may combine inequalities as in the following example.

Example 1 ($k = 3, r = 4$).
Let $A = \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0, \alpha_1 \geq 3\alpha_3\}$. Then

$$W = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}, \quad \tilde{W} = \frac{1}{7} \cdot \begin{pmatrix} 7 & 7 & 7 & 0 \\ 1 & 8 & 5 & -1 \\ 2 & 2 & 3 & -2 \end{pmatrix}$$

and (3.5) reads

$$\begin{aligned} 7\delta_1(z_1) + \delta_2(z_2) + 2\delta_3(z_3) &\geq 0 \\ 7\delta_1(z_1) + 8\delta_2(z_2) + 2\delta_3(z_3) &\geq 0 \\ 7\delta_1(z_1) + 5\delta_2(z_2) + 3\delta_3(z_3) &\geq 0 \\ - \delta_2(z_2) - 2\delta_3(z_3) &\geq 0. \end{aligned}$$

Corollary 1 is in the spirit of Fishburn's (1964, 1965) "method of equating coefficients"; see also Kirkwood and Sarin (1985). There appears also a formal analogy of part of our approach to that of Pearman and Kmietowicz (1986). Our matrix $\tilde{W}$ in Lemma 1 plays a similar role as their matrix M. However, the meanings are quite different: the latter paper is about LPI on probabilities while we focus on information about utility coefficients.

For conical linear information, the polar cone can be computed according to the methods given in Mathies and Rubin (1980). However, solving the linear program (3.1) directly seems to be the more efficient procedure.

4. Partial information on a multilinear utility

This section addresses the case of general multilinear utility functions (2.1) including the multiplicative type (2.5). We start with a numerical example. Then, the subsequent theorem parallels that of Sec. 3.

Example 2 ($k = 3$).
Let u be a general multilinear utility function for three realvalued attributes,

[2] I.e., all components of the product matrix are ≥ 0.

$$u(x_1, x_2, x_3) = \alpha_1 u_1(x_1) + \alpha_2 u_2(x_2) + \alpha_3 u_3(x_3) + \alpha_{12} u_1(x_1) u_2(x_2) + \alpha_{13} u_1(x_1) u_3(x_3) + \alpha_{23} u_2(x_2) u_3(x_3) + \alpha_{123} u_1(x_1) u_2(x_2) u_3(x_3)$$

and let some partial information on the coefficients of a standard multilinear decomposition of u be given by

$$A = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{123}) | \alpha_1, \alpha_2, \alpha_3 > 0; \alpha_{12}, \alpha_{13}, \alpha_{23} \leq 0; \alpha_{123} \geq 0\}.$$

We want to compare the following alternatives Y, X, and $\hat{X}$, each of which is capable of three possible three-attribute outcomes.

$$F_Y: \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ (800, 900, 700) & (800, 300, 400) & (400, 900, 400) \end{bmatrix}$$

$$F_X: \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ (600, 900, 700) & (400, 300, 400) & (800, 500, 700) \end{bmatrix}$$

$$F_{\hat{X}}: \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ (600, 900, 700) & (400, 300, 400) & (800, 500, 700) \end{bmatrix}$$

Theorem 2 (General multilinear case). *Assume that the utility has standard multilinear decomposition with $u_i(a_i) = 0$, $u_i(b_i) = 1$ for two fixed outcomes $(a_1, \ldots a_k)$ and $(b_1, \ldots b_k)$, further that the univariate utilities are increasing and the coefficient vector $\alpha = (\alpha_I)_{\emptyset \neq I \subset K}$ belongs to some given set A, $\alpha_{\{i\}} > 0$ for all i. With respect to this information an alternative Y dominates an alternative X if and only if for all z*

$$\inf_{\alpha \in A} \sum_{\emptyset \neq I \subset K} \alpha_I \delta_I(z_I) \geq 0 \tag{4.1}$$

where

$$\delta_I(z_I) = P(Y_I \geq z_I) - P(X_I \geq z_I). \tag{4.2}$$

A proof of Theorem 2 is found in the Appendix. Now let A be a set of coefficient vectors α with $\alpha_I = 0$ whenever $|I| \geq 2$. Then, the utility function has an *a priori* additive decomposition, and (4.1) reduces to (3.1). Thus, Theorem 1 is a consequence of Theorem 2.

Concerning conical linear information, Lemma 1 and its proof carry over almost verbatim by letting a and $\delta(z)$ be vectors in $(2^k - 1)$-space and assuming that rank $W = 2^k - 1$. Here also negative signs of α_I can occur when $|I| \geq 2$. We do not repeat the details.

Instead of this we apply the modified lemma to Example 2: Here, W has format 7×7 with multiindexed rows and columns, $W = (w_{IJ})$ where $w_{IJ} = 1$ if $I = J \in \{1, 2, 3, 123\}$, $w_{IJ} = -1$ if $I = J \in \{12, 13, 23\}$, and $w_{IJ} = 0$ else. Obviously $(W^{-1})' = W^{-1} = W$, and (see the lemma) $\tilde{W} = W$. Thus the conditions on $\delta_I(z_I)$ are sign conditions corresponding to those on α_I given in the information; $\delta_I(z_I) \geq (\leq) 0$ whenever $\alpha_I \geq (\leq) 0$. There are at most twelve different choices of z in $\mathbb{R}^3$ where we have to check the sign of $\delta_I(z_I)$, $I \subset \{1, 2, 3\}$. When comparing $\hat{X}$ and Y, e.g., at $z_{123} = (600, 500, 700)$ we get $\delta_{123}(z_{123}) = P(Y \geq z_{123}) - P(\hat{X} \geq z_{123}) = 0.5 - 0.75 < 0$, whereas at $z_{123} = (800, 900, 700)$ we get $\delta_{123}(z_{123}) = 0.5 - 0.0 > 0$; therefore with the set A given above neither Y dominates $\hat{X}$ nor viceversa. Further, it can be seen that no one of the three alternatives X, $\hat{X}$, and Y dominates another one. Alternatively, if $\tilde{A} = \{(\alpha_I)_{I \subset \{1,2,3\}} | \alpha_1, \alpha_2, \alpha_3 > 0, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{123} \geq 0\}$ we get $\delta_I(z_I) \geq 0$ for all I. This leads to dominance of Y over X. Further, $\hat{X}$ dominates X with respect to $\tilde{A}$, but there is no dominance between Y and $\hat{X}$.

Example 3 ($k = 2$).

Assume that u is multilinear with $k = 2$ and that $A = \{(\alpha_1, \alpha_2, \alpha_{12} | \alpha_1 \geq 3\alpha_2 > 0, \quad \alpha_1 \leq 3\alpha_{12}\}$. Then $\tilde{W} = (W')^{-1}$,

$$W = \begin{pmatrix} 1 & 0 & -1 \\ -3 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \tilde{W} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1 & 1/3 \end{pmatrix}.$$

With

$$F_X: \begin{bmatrix} 0.75 & 0.25 \\ (6, 9) & (4, 3) \end{bmatrix}, \quad F_Y: \begin{bmatrix} 0.5 & 0.5 \\ (4, 9) & (6, 3) \end{bmatrix}$$

we check $\delta_I(z_I)$ at $z = (6, 9)$, $(4, 9)$, $(6, 3)$, and $(4, 3)$. From Table 1 we see immediately that X dominates Y. Note that X stochastically dominates Y in each of its attributes.

Table 1

z	$(6, 9)$	$(4, 9)$	$(6, 3)$	$(4, 3)$
$\delta_1(z_1)$	0.25	0.0	0.25	0.0
$\delta_2(z_2)$	0.25	0.25	0.0	0.0
$\delta_{12}(z_{12})$	0.75	0.25	0.25	0.0

Now, the multiplicative case of utility decomposition is treated. As we know from the above, it is a special case of multilinear decomposition having less coefficients, viz. $\alpha_1, \alpha_2, \ldots \alpha_k$, and β. Hence, Theorem 2 applies with an arbitrary information on these coefficients. Two important examples of information on signs of coefficients are presented as a corollary of Theorem 2:

Corollary 2 (Multiplicative case). *If the utility function has a standard multiplicative decomposition with increasing univariate utilities and the coefficients are restricted as follows, then an alternative Y dominates an alternative X if and only if for every z*

(i) $P(Y \geq z) - P(X \geq z) \geq 0$ holds, provided that $\alpha_i \geq 0$ for $i = 1, 2, \ldots k$ and $\beta > 0$,

(ii) $P(Y \leq z) - P(X \leq z) \leq 0$ holds, provided that $\alpha_i \geq 0$ for $i = 1, 2, \ldots k$ and $\beta < 0$.

In order to derive this corollary from Theorem 2, recall that for a multiplicative utility decomposition $\alpha_I = \beta^{|I|-1} \prod_{i \in I} \alpha_i$ holds. The details of proof are omitted.

For Corollary 2 and related results see Mosler (1984) and Scarsini (1988). Other conical linear information on coefficients of multilinear utility decompositions which involves notions of multiattribute risk posture is discussed in Mosler (1987). The multiplicative case with known univariate utilities is treated in Hazen (1986).

5. Conclusions

When a multilinear utility function is only known to have increasing univariate utilities and scaling coefficients in some given set, conditions have been derived for an alternative to dominate another one. The general multilinear form of an utility function includes the additive and multiplicative cases, and the arbitrary information on scaling coefficients includes linear partial information (LPI) and conical LPI. LPI and conical LPI are easily assessed by standard procedures. In a number of important cases of LPI, closed form results of the conditions have been derived.

The findings of Hazen (1986) and previous authors have been modified and extended in several respects: instead of known univariate utilities we have employed unknown increasing ones, and instead of additive/multiplicative utilities we have considered general multilinear ones. Further, we have introduced a much more general prior information about the scaling coefficients.

We close this paper with some remarks on decision methodology. As a procedure for practical decision making, the above approach needs the following kinds of information (besides the list of alternatives): First, some information on utility independence of attributes (or on similar properties of the preference) which is standard in multiattribute decision methodology and allows for a proper multilinear, multiplicative or additive decomposition, see e.g., Keeney and Raiffa (1976). Second, the information that the univariate utilities are increasing (after an appropriate reordering of attribute levels). Third, after having fixed two outcomes a and b, some partial information on the coefficients; this information may be general, linear, or conical linear.

The procedure consists of several steps: (1) Assess alternatives, their possible outcomes, and the probabilities of their possible outcomes. (2) Assess utility independence properties and decompose the utility function; see Keeney and Raiffa (1976). (3) Assess increasingness of univariate utilities. (4) Fix outcomes a and b, and assess partial information on coefficients. (5) Check pairs of alternatives for dominance (by use of the results on conical linear information or by solving the program directly). Remove inefficient alternatives (by use of some partial ranking algorithm as given, e.g., in Kirkwood and Sarin 1985, or Bawa et al. 1979).

The computational feasibility, of course, depends on the number of attributes k and the number of possible levels of attributes. However, in view of the standard MAUT applications ($k \leq 4$), the limitations of our approach are not computational ones, but rather the usual limitations of MAUT: knowing all alternatives, outcomes, attributes, probabilities, assuming the expected utility hypothesis, and assessing utility independence.

A final remark, concerning the practicability of the approach, viz. the sizes of the efficient sets resulting from given partial information: There is considerable empirical evidence in the literature that in many practical situations the ranking of alternatives does not depend heavily on the specific coefficients used in an additive utility function; see Schoemaker and Waid (1982) and the references therein. If this is also the case with a multilinear utility function (which has still to be investigated), applying our procedure to these situations will be likely to result in relatively small efficient sets.

Appendix

Proof of Theorem 2. Necessity: Let $z \in C = C_1 \times \ldots C_k$ and $\alpha \in A$. We show that (4.1) is tantamount to (2.7) with some particular $u = w_z$ having a standard multilinear decomposition

$$w_z(x) = \sum_{J \subset K} \alpha_J \prod_{i \in J} 1_{[z_i, \infty[}(x_i), \tag{A.1}$$

where 1_S denotes the indicator function of a set S, defined by $1_S(\xi) = 1$ if $\xi \in S$, $1_S(\xi) = 0$ if $\xi \notin S$. The above product over J is unity if and only if $x_i \geq z_i$ for every $i \in J$, else the product vanishes. Hence, $Ew_z(X) = \sum_{J \subset K} \alpha_J P(X_J \geq z_J)$ holds, and the same for $Ew_z(Y)$. If all $z_i > c_i$ (where c_i denotes the lower boundary of C_i), the decomposition (A. 1) is standard multilinear with $b_i = z_i$ and with some $a_i < z_i$, $\alpha \in A$. Therefore, if Y dominates X with respect to A it follows that $Ew_z(X) \leq Ew_z(Y)$; hence (4.1). If $z_i = c_i$ for some i, the decomposition is not standard multilinear. However, w_z is the pointwise dominated limit of such decompositions and by Lebesgue's convergence theorem we again conclude $Ew_z(X) \leq Ew_z(Y)$ and (4.1). Sufficiency: Let u have a standard multilinear decomposition with $\alpha \in A$ and assume first that all u_i are right continuous. Since u_i is increasing, $0 = u(a_i) \leq u_i(x_i) \leq u_i(b_i) = 1$, we may define a probability measure μ_i, $\mu_i([a_i, x_i]) = u_i(x_i)$, on C_i, $i \in K$. Let $\mu = \mu_1 \otimes \ldots \mu_k$ be the product measure on C. Then for any I,

$$\prod_{i \in I} u_i(x_i) = \mu(\{z \in C | z_i \leq x_i, i \in I\}) = \int_C g_I(z, x_I) \mathrm{d}\mu(z)$$

where

$$g_I(z, x_I) = \begin{cases} 1 & \text{if } x_I \geq z_I \\ 0 & \text{else} \end{cases}.$$

Then

$$Eg_I(z, X_I) = P(X_I \geq z_I),$$

$$Eu(X) = \sum_{I \subset K} \alpha_I E\left(\prod_{i \in I} u_i(X_i)\right)$$

$$= \sum \alpha_I E\left[\int_C g_I(z, X_I)\mathrm{d}\mu(z)\right]$$

$$= \int_C \sum \alpha_I E g_I(z, X_I)\mathrm{d}\mu(z)$$

$$= \int_C \sum \alpha_I P(X_I \geq z_I)\mathrm{d}\mu(z).$$

Similarly,

$$Eu(Y) = \int_C \sum \alpha_I P(Y_I \geq z_I)\mathrm{d}\mu(z).$$

Therefore, from (4.1) we conclude $Eu(Y) - Eu(X) \geq 0$. This proves the theorem for right continuous u_i. If some u_i are not right continuous, they may be approximated in a standard way by right continuous $u_i's$ to yield again $Eu(Y) - Eu(X) \geq 0$. QED

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