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ROBUST COMPARISONS OF SPATIAL PATTERNS

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A spatial allocation (a so-called pattern) is represented by a finite measure in two-space, and choices between alternative patterns are based on an average value over space. This paper presents rules for robust comparisons of patterns in the sense that one pattern should be uniformly better than another one with respect to all elements of a certain class of valuation functions. In particular, we investigate the optimum location of a new pattern (i) which interacts with a single given center and (ii) which interacts with several given patterns.

1. Introduction

Most spatial planning is long term planning. The establishment of public land uses, the construction of housing estates and the investment in a transportation system have in common that, once a certain allocation in space has been decided for, its consequences persist over many years.

On the other hand, decisions between alternative plans and the evaluation of different allocations are usually based on facts and quantities which are known for a short period of time only but which may considerably change during the rest of the planning period due to unforeseen effects.

Such unforeseen variations of the valuation basis can be effected, e.g., by exogene factors (like the rises of oil prices in the past decade which have caused shifts in transportation demands and technologies), by common changes of individual tastes (as concerning the desirability of housing in new suburbs) or by the allocation decision itself (like the improvement of a transportation system which may evoke some additional demand).

This paper presents some new ideas to cope with the uncertainty of the valuation basis, at least for the first two effects mentioned.

We shall envisage planning situations which are distinguished by geographical and/or institutional restrictions so that the number of feasible alternatives is small, and we shall attack the following problem. Find conditions under which one can say that a given allocation is better than a given competing allocation for a large range of evaluation data the choice could possibly be based upon. This is what we call a robust comparison; it is some kind of worst-case analysis. One allocation should be uniformly better than the other one with respect to a whole class of valuations which is supposed to include every actually occurring valuation. In other words, the first one should be better than the second one for 'any possible' valuation.

In what follows we give rules for the robust comparison of alternative spatial patterns with respect to certain classes of evaluation functions. In section 2 the general model is defined; in section 3 we treat the case of evaluation according to the level of aggregate interactions with a single center, and give simple sufficient conditions for several classes of evaluations. In section 4 optimal robust allocations of a pattern with respect to several given patterns are characterized when the evaluation functions are convex resp. concave: In these cases it comes out that the search of an optimal pattern can be restricted to one-point-patterns or patterns of extreme points, respectively.

2. The basic model

Let K be a (Borel) set in the real plane. By a *spatial pattern* in K we mean a finite measure on K, i.e., a non-negative, countably additive function being defined on the (Borel) subsets of K; different spatial patterns are denoted by μ and ν.[1] Note that the definition includes discretely and continuously distributed patterns as well as 'mixed' distributions. We assume that a spatial pattern μ in K is evaluated according to its *total value*

$$\mu(p) = \int_K p(x)\mu(\mathrm{d}x), \tag{1}$$

where p is a measurable function from K into the reals. For each location $x \in K$ $p(x)$ indicates the resulting value — payoff, utility, or negative cost — when a unit of mass is allocated at x.

This basic setting fits to numerous planning situations in which a central agency has to decide between alternative spatial allocations of activities and in which decisions are based on some aggregate or average value and no economies of scale are present. Examples of allocations are: locating an industry, choosing a market area, investing in a transportation system, dispersing obnoxious activities, and establishing public land uses. In these five cases $p(x)$ may be introduced as a negative cost of production and transportation, a competitive strength, a negative transportation cost, a negative neighborhood cost, or a degree of accessibility, respectively.

In the applications mentioned, information on the type of p can often be derived from theoretical assumptions — like an underlying law of gravity or an intervening opportunity hypothesis in interaction models — while the value of certain parameters remain to be determined from the actual data.

In the sequel we investigate conditions in terms of μ and ν under which

[1] For an extensive treatment of this approach, see Faden (1977).

the total value of pattern μ is uniformly larger than the total value of a second pattern v with respect to all p from a given class P of value functions. The conditions to be presented below correspond to conditions for stochastic dominance of risky alternatives as have been established in the recent literature on decisions under risk [Brumelle and Vickson (1975), Mosler (1982)]. Concrete applications to given alternative spatial patterns are possible and may result in robust decisions between the patterns.

Let v and μ be given spatial patterns in K and let P be a set of measurable functions $K \to \mathbb{R}$. v is said to be *uniformly larger* than μ for $p \in P$ — shortly, v is *P-larger* than μ — if

$$\mu(p) \leqq v(p) \qquad \text{for all } p \in P \tag{2}$$

holds as far as the integrals on both sides of the inequality exist and are finite.

In a given family of patterns, condition (2) defines a relation — namely a semi-ordering — according to which, in general, some pairs of patterns are comparable but some others are not; any case, by comparing each pair of patterns in the family an efficient set of patterns can be constructed which excludes all uniformly smaller patterns.

3. Spatial interaction with a center

Consider $p(x)$ as the unit value of some aggregate interaction between an activity to be located at x and a given 'center' fixed at 0. Assume that $p(x)$ depends only on the geometric distance from 0 to x which is measured by a norm $||x||$, $p(x) = q(||x||)$; thus, the total value (1) of a pattern μ becomes

$$\tilde{\mu}(q) = \int_K q(||x||)\mu(\mathrm{d}x). \tag{3}$$

When q is an increasing function, it is called an *economic distance*; when it is decreasing, q is interpreted as a *law of interaction*. As the negative of an economic distance is a law of interaction we will investigate only the latter one. Simple choices of decreasing q that have been frequently employed in the literature are

$$q(s) = ae^{-bs} + c, \tag{4}$$

$$q(s) = (a/s^b) + c, \tag{5}$$

for $s > 0$, where $a \geqq 0$, $b > 0$, and $c \in \mathbb{R}$ are constants. (4) and (5) are known as the *negative exponential law* and the *gravity law* of interaction, respectively.

Alternative patterns shall be compared according to aggregate interaction with the center or, in other terms, according to the centrality of the center with respect to the patterns. Thus, $\tilde{\mu}(q)$ represents an *index of centrality*.

To illustrate this we touch three applications: the first one is a problem in marketing, namely the spatial allocation of an advertising budget. Suppose that an urban area is covered by a — typically small — number of advertising media in which a shopping center wants to spend its advertising budget. Let μ and ν represent two distributions of potential consumers being reached by ads under alternative allocations of the budget, and let allocation decisions be based on aggregate consumer retail expenditures. $q(s)$ may denote the retail expenditures of a consumer who lives at distance s from the shopping center. A gravity law (5) is used for q, e.g., in Vorhees/Lakshmanan (1966); in this case we are interested in decisions which are robust with respect to the parameters a, b, and c in (5).

The second illustration is lent from transportation planning. Consider two alternative investments into a system of transport to a center which result in different individual transportation costs, say, $f(x)$ and $g(x)$ for a unit located at point x in the area K. Assume that at location x the demand for transportation to the center depends mainly on the geometric distance $s=||x||$ away from the center and let $q(s)$ denote a proper density of demand. If q is subject to unexpected changes due to the varying attraction of the center as well as due to induction effects, we may assume q to remain a decreasing and convex function of s. Therefore, total transportation costs

$$\int_K q(||x||)f(x)\,\mathrm{d}x \quad \text{and} \quad \int_K q(||x||)g(x)\,\mathrm{d}x$$

should be compared with respect to all such functions.

Third, we want to compare alternative locations of housing projects when an air polluting source is present. Our evaluation criterion is total damage to human health caused by the polluter. According to a formula of atmospheric diffusion [cf. Pasquill (1962)] the concentration of gases in the air is inverse proportional to direct distance from the source. As for most air pollutants the relations between concentration inhaled and damage to human health are not exactly known, a reasonable assumption takes the damage q to be a non-decreasing and convex function in concentration. We conclude that q is decreasing and convex in distance.

For convenience of notation we restrict the discussion to the Euclidean norm and to patterns μ which possess a density $\tilde{f}$ in $\mathbb{R}^2$; other norms like the rectilinear norm or the supremum norm as well as non-continuous patterns can be treated similarly. With $\tilde{f}$ being given in polar coordinates (s, ϕ), the total value (3) of μ reads

$$\int_0^\infty \int_0^{2\pi} q(s)\tilde{f}(s,\phi)s\,\mathrm{d}\phi\,\mathrm{d}s, \tag{6}$$

which equals

$$\int_0^\infty q(s)f(s)\,\mathrm{d}s, \tag{7}$$

where the abbreviation

$$f(s)=\int_0^{2\pi} \bar{f}(s,\phi)s\,\mathrm{d}\phi \tag{8}$$

is used.

Now, let us compare two given spatial patterns μ and v under the premises that we do not know their (common) interaction law q but only a class Q which includes q.

v is *Q-larger* than μ if

$$\int_0^\infty q(s)f(s)\,\mathrm{d}s \leqq \int_0^\infty q(s)g(s)\,\mathrm{d}s \tag{9}$$

for all $q\in Q$ (as far as the integrals exist and are finite) where g is derived from v in the same way as f from μ.

When $\mu(\mathrm{K})=v(\mathrm{K})$ and q is known to be of negative exponential type (4) while the levels of the constants a, b and c are indeterminate, this condition means that

$$\int_0^\infty e^{-bs}f(s)\,\mathrm{d}s \leqq \int_0^\infty e^{-bs}g(s)\,\mathrm{d}s \tag{10}$$

should hold for all $b>0$ or, in other words, the Laplace transform of g should exceed that of f. If some a priori knowledge is available concerning the value of b so that $b\in B$, and B is known, of course (10) should be considered for $b\in B$ only.

If (10) applies for all $b>0$, it can be shown[2] that the inequality (9) does not hold only for all q of type (4) but also for all q which are *completely monotone functions*, i.e., which possess derivatives of arbitrary order alternating in sign so that $q'\leqq 0$, $q''\geqq 0$, $q'''\leqq 0,\dots$ etc. Particularly, as any gravity law (5) is completely monotone, we get the following proposition:

P.1. If a pattern is uniformly larger than another pattern for all negative exponential laws then it is uniformly larger for all gravity laws.

[2]According to a theorem by Bernstein [cf. Phelps (1965, p. 11)] every completely monotone function q can be written $q(s)=\int_0^\infty e^{-bs}\lambda(\mathrm{d}b)$, with a unique Borel measure λ on $\mathbb{R}_+$. Hence, (9) follows from (10) by inserting this equation and exchanging the order of integration.

More flexible forms of the interaction laws (4) and (5) are at hand when the constant a is replaced by a weight $a(s) \geqq 0$ depending on distance s. Then, if the weight is a non-increasing and convex function of distance, the same applies to q. More generally, we want to compare spatial patterns for arbitrary non-increasing and convex laws of interaction. With the definition

$$H(t) = \int_0^t [f(s) - g(s)]\, \mathrm{d}s \quad \text{for} \quad t \geqq 0, \tag{11}$$

there holds:[3]

P.2. ν is uniformly larger than μ for all q which are non-increasing and convex if and only if

$$\int_0^r H(t)\, \mathrm{d}t \leqq 0 \quad \textit{for all} \quad r \geqq 0. \tag{12}$$

Freight rates are, as a rule, non-decreasing and concave in distance, hence their negative is non-increasing and convex; so the preceding proposition applies to comparisons which are based on total shipment freight to a center. In other circumstances when the shipment of perishable goods is involved negative transportation costs may be non-increasing and concave due to the growing rate of goods perished. To this setting a related proportion applies:

P.3. Assume $\mu(K) = \nu(K)$. Then ν is uniformly larger than μ for all q which are non-increasing and concave if and only if

$$\int_r^\infty H(t)\, \mathrm{d}t \leqq 0 \quad \textit{for all} \quad r \geqq 0. \tag{13}$$

Analogous propositions hold for q being non-decreasing and concave (respectively non-decreasing and convex) where inequality (12) [respectively (13)] is replaced by the reversed one; this is proved by changing q into $-q$.

In order to check (12) and (13) there exist simple sufficient criteria concerning the sign of H or the sign of $h = f - g$, the first of which are illustrated in fig. 1:

P.4. Assume that $H(r)$ changes sign at most once and that $\int_0^\infty H(r)\, \mathrm{d}r \leqq 0$. If $H(r)$ starts with a negative (resp. positive) sign then (12) [resp. (13)] holds.

P.5. Assume that $h(s) = f(s) - g(s)$ changes sign at most twice and $\int_0^\infty sh(s)\, \mathrm{d}s \leqq 0$. If $h(s)$ starts with a negative (resp. positive) sign then (12) [resp. (13)] holds.

[3]The proposition corresponds to the second order stochastic dominance theorem; see Brumelle and Vickson (1975).

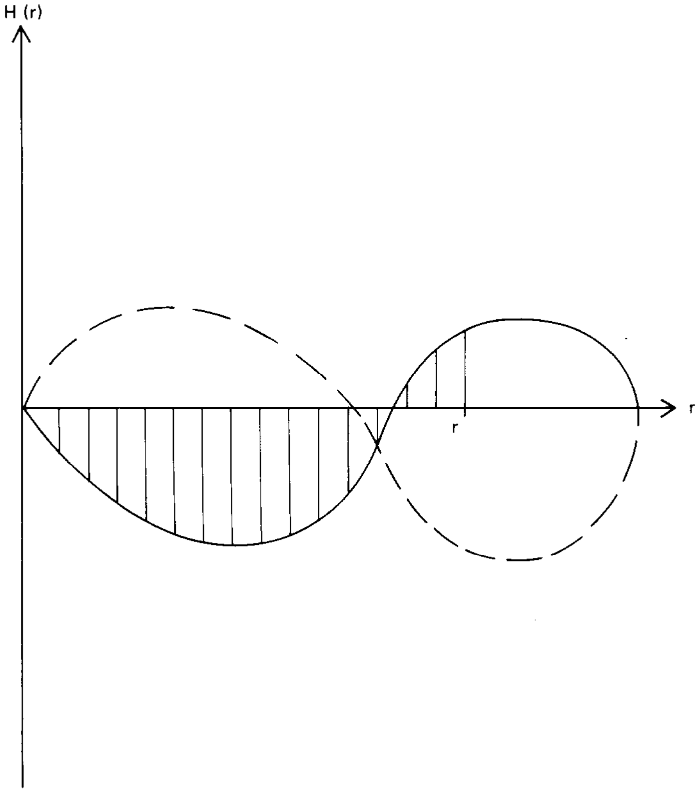

Fig. 1. Illustration of criteria concerning the sign of $H(r)$, sufficient for (12) ______, sufficient for (13) _ _ _ _.

It is not difficult to show that the conditions stated in the latter proposition imply the conditions stated in the preceding one. So, the second proposition follows from the first.

Now, let us return to condition (10) above that applies to completely monotone functions. As any completely monotone function is non-increasing and convex it comes out that condition (12) implies condition (10). Further conditions for robust comparisons of patterns can be derived which operate 'between' (10) and (12); this may be done along the lines of Rolski (1976) and Mosler (1982, sec. 7). — Finally, we may compare two patterns with respect to all possible laws of interaction; there holds a simple proposition:

P.6. Assume $\mu(K)=\nu(K)$. ν is uniformly larger than μ for all non-increasing q if and only if

$$H(t) \leqq 0 \quad \text{for all} \quad t \geqq 0. \tag{14}$$

The proposition can be proved like theorem 2.1 in Brumelle and Vickson (1975). Obviously, (14) implies (13) and (12).

For the rest of this section, we drop the above restriction that the law of interaction is a function of distance alone. For every $x \in K$ let $p(x)$ denote the level of interaction between x and the center 0. In some spatial interaction models (cf. e.g., the empirical findings on transportation costs by Palander (1935, p. 308 f) it may be assumed that the loci of equal interaction — i.e., the isocurves of p — enclose convex regions of K; it follows that p is a quasiconcave function in x having its maximum at 0. The following theorem the proof[4] of which is omitted gives a necessary and sufficient condition for a comparison to be robust with respect to all such functions.

Theorem 1. Let μ and ν be patterns in K of equal mass $\mu(K)=\nu(K)$. ν is P-larger than μ for

$$P=\{p:K\to\mathbb{R}\,|\,p \text{ quasiconcave, } p(0)\geqq p(x) \text{ for all } x\} \tag{15}$$

if and only if $\mu(C)\leqq\nu(C)$ for all $C\subset K$ which are convex and contain the origin.

We remark that the assumption of equal total mass in the theorem may be dropped when only those functions are considered which are bounded below by a common bound γ, i.e., when P is substituted by the class

$$P_\gamma=\{p\in P\,|\,p(x)\geqq\gamma \text{ for all } x\}.$$

4. Location of a new pattern with respect to given patterns

In this section we investigate the quite general situation that a finite number of patterns is taken as given and a new pattern is to be located according to some total value of interaction. A simple example is given by the classical Steiner–Weber location problem where all patterns considered are one-point-measures.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be existing patterns in K, and μ a pattern to be located in K. The problem is stated in terms of interaction costs: $s_j(x,y)$ denotes the cost of interaction between a unit of μ being located at x and a unit of λ_j being located at y; $x,y\in K$, $j=1,2,\ldots,m$. Find a measure μ on K of given total mass, say $\mu(K)=1$, which solves the following minimization problem:

$$\int_K \sum_{j=1}^{m} \int_K s_j(x,y)\,\lambda_j(\mathrm{d}y)\,\mu(\mathrm{d}x)\to\min_{\mu}. \tag{16}$$

[4]It can be deduced from a slight generalization of theorem 3.13 in Mosler (1982).

Note that the integrand

$$p(x)=\sum_{j=1}^{m}\int_{K} s_j(x,y)\lambda_j(\mathrm{d}y) \tag{17}$$

of the μ-integral is a convex (resp. concave) function in x if for every j and y $s_j(\cdot,y)$ is a convex (resp. concave) function.

For each j the value $s_j(x,y)$ may be interpreted as a cost of transportation or a distance between x and y; when the distance is derived from a norm, $s_j(x,y)$ is convex in x for every y. In the classical Steiner–Weber problem λ_j is a one-point-measure and $s_j(x,y)=w_j\|x-y\|$ with weights $w_j \geqq 0, j=1,2,\ldots,m$. In commuting traffic the individual cost of transportation (mainly cost of time) is likely to be convexly increasing in distance; then again, $s_j(\cdot,y)$ is a convex function.

To give another interpretation, $s_j(x,y)$ may denote a cost of proximity, e.g., the concentration of a pollutant at x per unit of pollution source located at y. If the concentration is linearly decreasing in distance and if the distance is based on a norm, $s_j(x,y)$ is concave in x for every j and y.

The recent literature on decision theory under risk[5] contains conditions for the comparison of random vectors with respect to certain classes of multivariate utility functions which may be transposed to the present context. They yield rules by which pairs of patterns with multivariate value functions can be compared in a robust way. We will not go through these rules here but only state a theorem on the agglomeration (or dispersion) of a new pattern, the value function of which is convex (resp. concave).

Concerning the location of industries a common observation says: An industry, which is bound to a single source of raw material R and to a single market M, shows a tendency to locate either at the site of the market or at the site of the material, while intermediate locations are rare (and mostly due to exceptional transhipment facilities like free ports).

Tord Palander (1935, p. 310f) has attributed this tendency to peculiarities of the transportation system: terminal costs, and the gradient of freight tariffs being decreasing, so the total transportation costs become a concave function of location in the interval $[R,M]$ and are minimum at one of the boundary points R and M; see fig. 2.

Palander's analysis immediately carries over to the problem of locating a single plant when an arbitrary number of market and material sites is given being situated along a transportation line (e.g., a railway line); if, again, the transportation costs of each commodity are concave in distance an optimal location of the plant can be found at one of the sites given.

We extend this latter result in two directions: first, the restriction to one dimension is dropped and, second, instead of a single plant a whole industry,

[5]Cf. Mosler (1982, sect. 8–11).

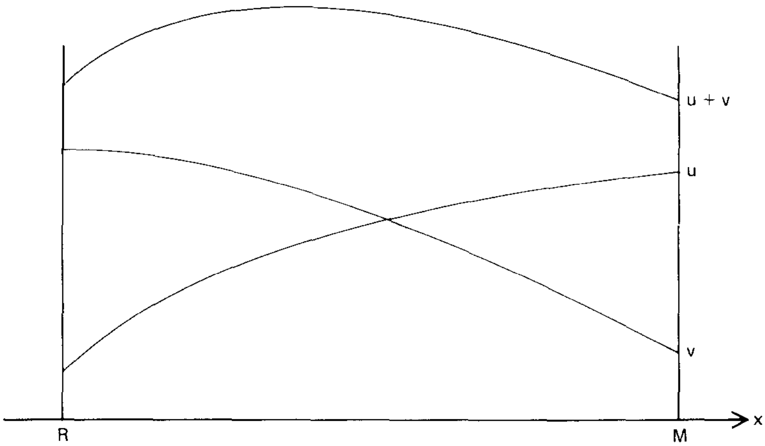

Fig. 2. Location of a single plant between a market M and a site of raw material R. Transportation costs of goods produced: v, resp. of raw material: u, depending on location: x.

i.e., a pattern of plants, is located. Note that a plant is a special case of an industry, namely a pattern the total mass of which is concentrated at a single point.

Let K be a subset of the real plane being convex, closed and bounded, e.g., the Euclidean unit disc or a non-degenerate convex polygon. Let $E(K)$ denote the set of extreme points in K, i.e., of those $x \in K$ which cannot be represented by a non-trivial convex combination of points in K,

$$E(K) = \{x \in K \mid x \neq ay + (1-a)z \text{ for all } a \in]0,1[\text{ and all } y, z \in K \text{ besides } y = z = x\}$$

E.g., when K is a polygon, $E(K)$ consists of its nodes; when K is the Euclidean disc, $E(K)$ is the unit sphere.

Theorem 2. Assume that there exists an optimal solution of (16).

(*i*) *If for every $y \in K$ and $j \in \{1, 2, \ldots, m\}$ $s_j(\cdot, y)$ is concave and continuous, then (16) has an optimal solution μ^* so that $\mu^*(E(K)) = \mu^*(K)$.*
(*ii*) *If for every $y \in K$ and $j \in \{1, 2, \ldots, m\}$ $s_j(\cdot, y)$ is convex and continuous, then (16) has an optimal solution μ^* so that $\mu^*(\{x^0\}) = \mu^*(K)$ for some $x^0 \in K$.*

Outline of proof. Let μ_0^* be an optimal solution and

$$r^* = \int_K x \mu_0^*(\mathrm{d}x)$$

its barycenter. Let p be defined by (17).

(i) By Choquet's theorem [cf. Phelps (1965)] there exists a measure μ^* so that $\mu^*(E(K))=\mu^*(K)=1$ and $\mu^*(v)\leqq\mu_0^*(v)$ for all functions v which are concave and continuous. As the functions $s_j(\cdot,y)$ are concave and continuous, the same holds for p. Therefore μ^* is not worse than μ_0^*, hence μ^* itself is an optimal solution.

(ii) Let μ^* be the pattern which assigns unit mass to r^*. As the $s_j(\cdot,y)$ are convex and continuous, the same holds for p.

Therefore,

$$\int_K p(x)\mu^*(\mathrm{d}x) = p(r^*) = p\left(\int_K x\mu_0^*(\mathrm{d}x)\right) \leqq \int_K p(x)\mu_0^*(\mathrm{d}x)$$

by Jensen's inequality. We conclude that μ^* is an optimal solution, too. Q.E.D.

The theorem tells that in the case of concave $s_j(\cdot,y)$ the search for an optimal pattern in K can be restricted to those patterns the total mass of which is concentrated on the extreme points of K; in the case of convex $s_j(\cdot,y)$ the search should cover only one-point-patterns. To give a simple application, it follows from the theorem that in a Steiner–Weber industry location problem nothing can be gained by allowing the new industry to disperse into several sites in K.

The theorem is valid not only for K in two-dimensional space but also for higher dimensions and in quite general abstract spaces [cf. Phelps (1965) for weaker premises to the Choquet theorem]. So, the theorem extends to comparisons of space/time allocations as well as to other more abstract allocation problems.

References

Brumelle, S.L. and R.G. Vickson, 1975, A unified approach to stochastic dominance, in: W.T. Ziemba and R.G. Vickson, eds., Stochastic optimization models in finance (Academic Press, New York).

Faden, A.M., 1977, Economics of space and time (The Iowa State University Press, Ames, IA).

Mosler, K.C., 1982, Entscheidungsregeln bei Risiko: Multivariate stochastische Dominanz (Springer, Berlin-Heidelberg-New York).

Palander, T., 1935, Beiträge zur Standorttheorie (Almqvist Wiksell, Uppsala).

Pasquill, F., 1962, Atmospheric diffusion (Van Nostrand, London).

Phelps, R.R., 1965, Lectures on Choquet's theorem (Van Nostrand, Princeton, NJ).

Rolski, T., 1976, Order relations in the set of probability distribution functions and their applications in queuing theory (Diss. Math., Warsaw).

Vorhees, A.M. and T.R. Lakshmanan, 1966, A market potential model and its application on planning regional shopping centers, in: R.M. Haas, ed., Proceedings of the 1966 Fall Conference of the American Marketing Association, Chicago, IL, 831–845.