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STOCHASTIC DOMINANCE DECISION RULES WHEN THE ATTRIBUTES ARE UTILITY INDEPENDENT*

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In multivariate decisions under risk, assessing the complete utility function can be a major obstacle. Decision rules are investigated which characterize uniformly better alternatives with respect to a whole class of utility functions. In this paper independence assumptions are imposed on the preference structure while the levels of attributes may be stochastically dependent in an arbitrary way.

The utilities considered are additive, multiplicative, or multilinear. Necessary and sufficient conditions are developed for uniform decisions over utilities with common substitutional structure and where the univariate conditional utilities show qualitative properties such as risk aversion. The rules are direct extensions of known univariate rules and easy to evaluate.
(MULTIATTRIBUTE DECISION; SEPARABLE UTILITY; RISK AVERSION)

Consider a decision problem having risky outcomes described by more than one, say n, attributes. When the probabilities of the outcomes are known and the decision is to be based on expected utility, the main practical difficulty arises in assessing the n-variate utility function u. In order to assess u, the decision making individual has to be asked many questions about his preferences on lotteries over multivariate outcomes, a procedure which in some practical situations appears to be a formidable or even impossible task. Provided certain separability properties of the utility function u can be established (this is the case when the attributes show some utility independence in the sense of Keeney and Raiffa 1976), assessing u is greatly simplified, but still n univariate utility functions and up to $2^n - 2$ constants have to be determined.

Therefore, there is a need for decision rules which do not require the complete knowledge of an individual's utility function but which employ only partial information about his preferences; especially, rules are needed which only use the knowledge of qualitative properties of the preference such as monotonicity or risk aversion. More precisely, given a qualitative property of u, we are looking for a decision rule which indicates that the first of two alternative prospects is not worse than the second one with respect to all u bearing this property. The decision rule shall consist in a condition concerning the alternative prospects; rules of this kind have been termed stochastic dominance rules in the literature; see the survey by Fishburn (1978).

Recent efforts have been made to extend the concept of stochastic dominance efficiency analysis from a single variable to the multivariate case, i.e., to develop conditions in terms of the probability distributions of two random vectors X and Y, which are necessary and sufficient for stochastic dominance (SD) of Y over X with respect to a given class of utilities. First investigations assumed either stochastically independent attributes (Levy 1973) or additively separable utilities or univariate utilities being defined on terminal wealth (Levy and Paroush 1974a) in which three cases the problem reduces to a univariate one. Huang et al. (1978a, b) presented the corresponding results on classes of utilities which show conditional risk aversion or DARA. Methods which are essentially multivariate have been introduced by Levy and Paroush (1974b) and Hadar and Russell (1974) who explored the relationship between utilities with signed partial derivatives and point by point dominance of probability distribution functions. Levhari, Paroush, and Peleg (1975) rediscovered (cf. Lehmann 1955) necessary and sufficient conditions for SD with respect to the class of general utilities as well as that of quasi-concave utilities. An alternative condition which is only sufficient for SD with general utilities has been given in Huang et al. (1978a).

In this paper independence assumptions are imposed on the preference structure while the levels of attributes may be stochastically dependent in an arbitrary way. We investigate multivariate SD with respect to classes of utilities for which the conditional utility of some bundles of attributes does not depend on the levels of the other attributes, which implies that the utilities considered are additive, multiplicative, or multilinear. Necessary and sufficient conditions are developed for SD over utilities with common substitutional structure and where the univariate conditional utilities show qualitative properties such as risk aversion. The rules are direct extensions of known univariate rules and easy to evaluate.

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Our aim is twofold: first, to unify and extend the notions of mulitvariate SD given in the literature and to provide a deeper insight into the differences as well as the similarities between the univariate and the multivariate cases; second, to open a practical alternative to the direct assessment of multivariate utilities via assessing univariate utilities as it has been widely propagated.

The paper is organized as follows: §1 introduces the definitions of SD and of utility independence, §2 reviews the additive case, §3 gives conditions for the multiplicative case which are proved in §4. §5 extends the conditions to multilinear utilities with given substitutional structure while §6 provides some conclusions.

1. Multivariate Stochastic Dominance and Utility Independence

Consider a set of attributes $\mathscr{A} = \{A_1, \dots A_n\}$ which attain real-valued levels $a_1, \dots a_n \in \mathbb{R}$. Let a decision maker (DM) be faced with an alternative of choosing one of two possible actions under uncertainty; if he chooses the first, attribute A_i will attain level X_i for $i = 1, \dots n$; if he chooses the second, A_i will attain Y_i for $i = 1, \dots n$. Here $(X_1, \dots X_n) = X$ and $(Y_1, \dots Y_n) = Y$ denote arbitrary random vectors in $\mathbb{R}^n$, originating from a common probability space $(\Omega, \mathscr{S}, P)$. We assume that the decision maker maximizes expected utility, i.e. he does not prefer the first action, with payoff X, if and only if[1]

$$\Delta u := E\left[u(X) - u(Y)\right] \leqslant 0 \tag{1.1}$$

where u denotes his utility function. In this paper, all utility functions are assumed to be von Neumann–Morgenstern and nondecreasing. Obviously, (1.1) for some u implies (1.1) for all $v = \alpha + \beta u$ where α and β are real constants, $\beta > 0$. Whether (1.1) holds or not, for a given u, depends on the marginal distributions of X and Y only. As an eventual stochastical dependence between X and Y does not affect the decision we do not specify their dependence structure.

When (1.1) holds uniformly for all u out of a given class U of utilities then Y is said to *dominate* X *with respect to* U; symbolically $X \leqslant_U Y$.

Next, consider a given nondecreasing preference ordering $\preccurlyeq$ of lotteries over $\mathbb{R}^n$ and a bounded utility function u which represents $\preccurlyeq$. A subset $\mathscr{A}' \subset \mathscr{A}$ of attributes is *utility independent* of its complement $\mathscr{A} \setminus \mathscr{A}'$ if preference between two lotteries which differ in levels of attributes $\in \mathscr{A}'$ only does not depend on the levels at which the attributes $\in \mathscr{A} \setminus \mathscr{A}'$ are held fixed. A well-known theorem (Keeney and Raiffa 1976, pp. 289ff) states that every subset $\mathscr{A}' \subset \mathscr{A}$ is utility independent of its complement if and only if there are functions $u^i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$, such that one of the following three representations applies: With $x = (x_1, x_2, \dots, x_n)$

$$u(x) = \alpha + \beta \sum_{i=1}^{n} u^i(x_i), \tag{1.2}$$

$$u(x) = \alpha - \beta \prod_{i=1}^{n} \left(-u^i(x_i)\right), \qquad u^i \leqslant 0, \tag{1.3}$$

$$u(x) = \alpha + \beta \prod_{i=1}^{n} u^i(x_i), \qquad u^i \geqslant 0, \tag{1.4}$$

where $\alpha, \beta \in \mathbb{R}$, $\beta > 0$. For alternative independence conditions to be imposed on the preference ordering as well as for procedures to verify them in real life, see Keeney and Raiffa (1976, Chapter 6).

Which of the three representations holds depends on substitutional properties of pairs of attributes. For any two attributes, A_j and A_k, $a = (a_1, \dots a_n) \in \mathbb{R}^n$, and any $b_j > a_j$, $b_k > a_k$, define a lottery $L_1(a, j, k, b_j, b_k)$ which gives levels (a_j, a_k) and (b_j, b_k)

[1] Conditions on expected values such as (1.1) are to be understood with the silent addition "if the expectation exists, possibly infinite".

to (A_j, A_k) with equal chance and define a lottery $L_2(a, j, k, b_j, b_k)$ which gives (A_j, A_k) levels (a_j, b_k) and (b_j, a_k) with equal chance, the levels of all other attributes A_i being fixed at a_i, $i \neq j, k$. With these definitions, (1.2) resp. (1.3) resp. (1.4) applies if and only if $L_1 \sim L_2$ resp. $L_1 \preccurlyeq L_2$ resp. $L_2 \preccurlyeq L_1$ for all j, k, a, $b_j > a_j$, $b_k > a_k$, and, in the latter cases, $L_1 \prec L_2$ resp. $L_2 \prec L_1$ for some j, k, a b_j, b_k; cf. Meyer (1977). $L_1 \prec L_2$ means that gaining either a higher level of A_j and a lower level of A_k or vice-versa is more desired than gaining either a lower or a higher level of both A_k and A_j. u is of type (1.3) when any two attributes are substitutional and the DM tends to diversify between them; u is of type (1.4) when any two attributes are complementary or the DM is risk seeking.

In the next three sections we will investigate stochastic dominance with respect to subsets of $U_0 := \{u : \mathbb{R}^n \to \mathbb{R} \mid u \text{ nondecreasing}\}$, namely, classes of utilities of type (1.2), (1.3), and (1.4), respectively, where the one-dimensional utilities u^i have certain properties. Note that if the u^i are concave functions then u in (1.2) is concave, too, whereas u in (1.3) and (1.4) is concave with respect to every argument x_i but in general not concave in x. Further, in (1.2) u is concave iff u is quasiconcave iff all u^i $(i = 1, \ldots n)$ are concave; see Cox (1973). In each case, u is nondecreasing if and only if all u^i are nondecreasing.

2. The Additive Case

This section briefly reviews conditions for stochastic dominance with respect to classes of additive utilities. Let $J_1, \ldots J_r$ be subsets of $\{1, \ldots n\} =: N$, $n_k := |J_k|$, and V^k sets of nondecreasing functions $\mathbb{R}^{n_k} \to \mathbb{R}$, $k = 1, \ldots r$. Assume that $\lambda v^k \in V^k$ whenever $v^k \in V^k$ and $\lambda > 0$, and that there exist functions $v^k \in V^k$ for which Δv^k is finite, $k = 1, \ldots r$. We consider SD with respect to the class of *additively decomposable* utilities[2]

$$W := \left\{ u \in U_0 \mid u(x) = \alpha + \beta \sum_{k=1}^{r} v^k(x_{J_k}),\ v^k \in V^k, \alpha \in \mathbb{R}, \beta > 0 \right\}.$$

THEOREM 1. *$X \preccurlyeq_W Y$ if and only if $X_{J_k} \preccurlyeq_{V^k} Y_{J_k}$ for all $k = 1, \ldots r$.*

PROOF. For any $u \in W$, $\Delta u = \sum_{k=1}^{r} \Delta v^k$. If $\Delta v^k \leqslant 0$ for all k then $\Delta u \leqslant 0$, which proves sufficiency. On the other hand, let $k \in \{1, \ldots n\}$ and $v^k \in V^k$. There exist $v^i \in V^i$ with Δv^i finite, $i \neq k$. Then

$$u_n := v^k + \frac{1}{n} \sum_{i \neq k} v^i \in W$$

for all $n \in \mathbb{N}$, $0 \geqslant \Delta u_n = \Delta v^k + n^{-1} \sum_{i \neq k} \Delta v^i$; therefore $0 \geqslant \Delta v^k$. Q.E.D.

The theorem yields an efficiency criterion for $\preccurlyeq_W$ whenever efficiency criteria for $\preccurlyeq_{V^k}$ are known.

Let us specialize the theorem by setting $J_k = \{k\}$ and $V^k = V$ for all k, and define

$$U_{aV} := \left\{ u \in U_0 \mid u(x) = \alpha + \beta \sum_{i=1}^{n} u^i(x_i),\ u^i \in V, \alpha \in \mathbb{R}, \beta > 0 \right\},$$

$$X \preccurlyeq_{aV} Y :\Leftrightarrow \Delta u \leqslant 0 \qquad \text{for all} \quad u \in U_{aV}.$$

COROLLARY 1 (Levy 1973, Huang, Vertinsky and Ziemba 1978b). *$X \preccurlyeq_{aV} Y$ if and only if $X_i \preccurlyeq_V Y_i$ for all $i = 1, \ldots n$.*

[2] For any $x \in \mathbb{R}^n$ and $J \subset \{1, \ldots n\}$ denote $x_J := (x_j)_{j \in J}$.

The most important applications of the corollary are obtained when V is specified as follows

$V_1 := \{v:\mathbb{R}\to\mathbb{R} \mid v \text{ nondecreasing}\}$,

$V_2 := \{v \in V_1 \mid v \text{ concave}\}$,

$V_3 := \{v \in V_2 \mid -v''/v' \text{ exists and is nonincreasing}\}$,

$V_4 := \{v \in V_2 \mid -v''/v' \text{ exists and } r_1 \leqslant -v''/v' \leqslant r_2\}$, where r_1 and r_2 are given real functions and the prime denotes derivation. Univariate efficiency criteria for $\leqslant_{V_k}$ are well known (cf. Levy 1973, Vickson 1977, and Meyer 1977). The sets U_{aV} are widely used classes of multivariate utility functions; an economic discussion can be found in Yaari (1978), and Meyer (1977).

Besides that, Theorem 1 may be applied to interdependent additive utilities such as bilateral dependent ones or to any fractional hypercube decompositions (see the review by Farquhar 1978), as well as to stationary (Koopmans 1960) and Markovian (Bell 1977) utilities.

3. The Multiplicative Case

In this section we develop efficiency criteria for stochastic dominance with respect to classes of multiplicative utilities; i.e. utilities of type (1.3) or (1.4). Define

$U_{m1} := \{u \in U_0 \mid u(x) = \alpha - \beta\prod_{i=1}^n(-u^i(x_i)),\ u^i \leqslant 0 \text{ nondecreasing}, \alpha \in \mathbb{R}, \beta > 0\}$,

$U_{m2} := \{u \in U_{m1} \mid u^i \text{ concave}\}$,

$U_{m3} := \{u \in U_0 \mid u(x) = \alpha + \beta\prod_{i=1}^n u^i(x_i),\ u^i \geqslant 0 \text{ nondecreasing}, \alpha \in \mathbb{R}, \beta > 0\}$,

$U_{m4} := \{u \in U_{m3} \mid u^i \text{ convex}\}$,

and for $j = 1,2,3,4: X \leqslant_{mj} Y :\Leftrightarrow \Delta u \leqslant 0$ for all $u \in U_{mj}$.

$u \in U_{m1}$ means that the DM considers the attributes as mutually utility independent and substitutional and that he/she tends to diversify between pairs of attributes. $u \in U_{m2}$ means that, in addition, the DM is risk aversive in any single attribute. On the other hand, $u \in U_{m3}$ corresponds to mutual utility independence together with attributes being complementary by pairs or together with a risk seeking DM. $u \in U_{m4}$ is, in addition, risk seeking in every single attribute.

We state four theorems which will be proved in the next section:

THEOREM 2. *$X \leqslant_{m1} Y$ if and only if $P(X \leqslant z) \geqslant P(Y \leqslant z)$ for all $z \in \mathbb{R}^n$.*

($\leqslant$ or $<$ between vectors means $\leqslant$ resp. $<$ between each of their components.) The theorem says that Y dominates X with respect to any utility function in U_{m1} if and only if the cumulative distribution function of Y does not exceed that of X. We see from the theorem that the dominance relation $\leqslant_{m1}$ coincides with that which Hadar and Russell (1974) define as "multivariate weak first degree stochastic dominance".

When $n = 1$, the condition of the theorem is a well-known criterion for SD with respect to all increasing utilities, i.e. for (univariate) first degree stochastic dominance; but for any higher dimension $n > 1$ the condition is necessary but not sufficient for SD with respect to all multivariate increasing utilities. The necessity is obvious; for a counterexample regarding sufficiency see below. The usefulness of Theorem 1 may be illustrated by the following simple corollary.

COROLLARY 2. *Let X and Y be identically distributed random vectors on $\mathbb{R}^n$, ϕ a function in $\mathbb{R}^n$ with $\phi(x) \geqslant x$ for all x; then $X \leqslant_{m1} \phi(Y)$. Especially, for any $a \in \mathbb{R}^n_+$, $X \leqslant_{m1} Y + a$.*

PROOF. For any $x, y \in \mathbb{R}^n$, $\phi(x) \leqslant y$ implies $x \leqslant y$; hence

$$\{\omega \mid \phi(Y(\omega)) \leqslant y\} = \{\omega \mid \phi(X(\omega)) \leqslant y\} \subset \{\omega \mid X(\omega) \leqslant y\},$$

therefore $P(\phi(Y) \leqslant y) \leqslant P(X \leqslant y)$ for any y. Q.E.D.

For the alternative relation $\leqslant_{m3}$ we get a closely related theorem and a corollary which parallels Corollary 2:

THEOREM 3. *$X \leqslant_{m3} Y$ if and only if $P(X > z) \leqslant P(Y > z)$ for all $z \in \mathbb{R}^n$.*

COROLLARY 3. *Let X and Y be identically distributed random vectors on $\mathbb{R}^n$, $\phi: \mathbb{R}^n \to \mathbb{R}^n$ with $\phi(x) \leqslant x$ for all x; then $\phi(Y) \leqslant_{m3} X$.*

The subsequent efficiency criteria for $\leqslant_{m2}$ and $\leqslant_{m4}$ involve integrals over $L_a := \{x \in \mathbb{R}^n | x \leqslant a\}$ and $R_a := \{x \in \mathbb{R}^n | x > a\}$, $a \in \mathbb{R}^n$.

THEOREM 4. *$X \leqslant_{m2} Y$ if and only if $\int_{L_a}[P(X \leqslant z) - P(Y \leqslant z)]dz \leqslant 0$ for all $a \in \mathbb{R}^n$.*

THEOREM 5. *$X \leqslant_{m4} Y$ if and only if $\int_{R_a}[P(X > z) - P(Y > z)]dz \leqslant 0$ for all $a \in \mathbb{R}^n$.*

When $n = 1$, Theorems 4 and 5 give the usual efficiency criteria for univariate second degree stochastic dominance.

We continue with several properties of the relations $\leqslant_{mj}$ which follow from Theorems 2 to 5.

When being applied to n-tuples of stochastically independent random variables, the criteria given in the theorems reduce to stochastic dominance conditions between univariate marginal distributions. In this case for all $a \in \mathbb{R}^n$

$$P(X \leqslant a) = \prod_{i=1}^{n} P(X_i \leqslant a_i) \quad \text{and} \quad \int_{L_a} P(X \leqslant z)\,dz = \prod_{i=1}^{n} \int_{-\infty}^{a_i} P(X_i \leqslant z_i)\,dz_i,$$

analogous equations hold for Y. From the equivalences

$$\prod_{i=1}^{n} P(X_i \leqslant a_i) \geqslant \prod_{i=1}^{n} P(Y_i \leqslant a_i) \qquad \forall a_1, a_2, \ldots a_n \in \mathbb{R}$$

$$\Leftrightarrow P(X_i \leqslant a_i) \geqslant P(Y_i \leqslant a_i) \quad \forall a_i \in \mathbb{R}, \forall i \in N, \text{ and}$$

$$\prod_{i=1}^{n} \int_{-\infty}^{a_i} P(X_i \leqslant z_i)\,dz_i \geqslant \prod_{i=1}^{n} \int_{-\infty}^{a_i} P(Y_i \leqslant z_i)\,dz_i \qquad \forall a_1, a_2, \ldots a_n \in \mathbb{R}$$

$$\Leftrightarrow \int_{-\infty}^{a_i} P(X_i \leqslant z_i)\,dz_i \geqslant \int_{-\infty}^{a_i} P(Y_i \leqslant z_i)\,dz_i \quad \forall a_i \in \mathbb{R}, \forall i \in N,$$

follows

$$X \leqslant_{m1} Y \Leftrightarrow P(X_i \leqslant a_i) \geqslant P(Y_i \leqslant a_i) \qquad \forall a_i \in \mathbb{R}, \quad \forall i \in N,$$

$$X \leqslant_{m2} Y \Leftrightarrow \int_{-\infty}^{a_i} P(X_i \leqslant z_i)\,dz_i \geqslant \int_{-\infty}^{a_i} P(Y_i \leqslant z_i)\,dz_i \qquad \forall a_i \in \mathbb{R}, \quad \forall i \in N;$$

hence, when X and Y both are stochastically independent random vectors, we have

$$X \leqslant_{m1} Y \Leftrightarrow X_i \leqslant_{m1} Y_i \qquad \text{for all} \quad i = 1, 2, \ldots n,$$

$$X \leqslant_{m2} Y \Leftrightarrow X_i \leqslant_{m2} Y_i \qquad \text{for all} \quad i = 1, 2, \ldots n.$$

Analogous criteria apply to $\leqslant_{m3}$ and $\leqslant_{m4}$.

COROLLARY 4. *For $j = 1, 2, 3, 4$ the following hold:*

(i) *$\leqslant_{mj}$ induces a semiordering in the set $\mathcal{F} := \{F: \mathbb{R}^n \to \mathbb{R} | F$ probability distribution function$\}$, i.e. $\leqslant_{mj}$ is a reflexive, transitive, and antisymmetric relation in $\mathcal{F}$.*

(ii) *For any $\{i_1, \ldots i_r\} \subset \{1, \ldots n\}$:*

$$(X_1, \ldots X_n) \leqslant_{mj} (Y_1, \ldots Y_n) \Rightarrow (X_{i_1}, \ldots X_{i_r}) \leqslant_{mj} (Y_{i_1}, \ldots Y_{i_r}).$$

PROOF. (i) Every SD relation is reflexive and transitive. The antisymmetry follows directly from Theorems 2 to 5.

(ii) For $j = 1$ and 3 this is immediately seen from Theorems 2 and 3, respectively. For $j = 2$ the assertion follows from Theorem 4 and the following lemma, the proof of which is omitted; for $j = 4$ analogously. Q.E.D.

LEMMA 1. *Assume* $\int_{L_b}[P(X \leqslant z) - P(Y \leqslant z)]\,dz \geqslant 0$ *for all* $b \in \mathbb{R}^n$; *let* $\{i_1, i_2, \dots, i_m\} \subset \{1, 2, \dots, n\}$ *with* $i_1 < i_2 < \cdots < i_m$, *and let* $a_{i_1}, \dots a_{i_m} \in \mathbb{R}$; *then*

$$\int_{-\infty}^{a_{i_m}} \cdots \int_{-\infty}^{a_{i_1}} \big[P(X_{i_\nu} \leqslant z_{i_\nu}\ \forall \nu = 1, \dots, m) - P(Y_{i_\nu} \leqslant z_{i_\nu}\ \forall \nu = 1, \dots, m)\big]\,dz_{i_1} \dots dz_{i_m} \geqslant 0.$$

There hold the obvious implications

$$X \leqslant_{m1} Y \Rightarrow X \leqslant_{m2} Y, \tag{3.1}$$

$$X \leqslant_{m3} Y \Rightarrow X \leqslant_{m4} Y, \tag{3.2}$$

$$X \leqslant_{U_0} Y \Rightarrow X \leqslant_{m1} Y \quad \text{and} \quad X \leqslant_{m3} Y, \tag{3.3}$$

but the reverse implications are not true. For (3.1) and (3.2) this is known even when $n = 1$; Levhari, Paroush and Peleg (1975) gave an example by which it can be seen that the reverse direction of (3.3) does not hold when $n = 2$ (hence, when $n \geqslant 2$).

Next, we investigate situations where some marginals of X and Y coincide. Let F_i and G_i denote the marginal distribution functions of X_i resp. Y_i, and $F_{\bar{i}}$ and $G_{\bar{i}}$ those of $X_{\bar{i}} := (X_1 \dots X_{i-1}, X_{i+1}, \dots X_n)$ resp. $Y_{\bar{i}} := (Y_1, \dots Y_{i-1}, Y_{i+1}, \dots Y_n)$. In general

$$X \leqslant_{m1} Y \quad \text{and} \quad Y \leqslant_{m3} X \Rightarrow F_i = G_i \qquad \text{for all} \quad i = 1, \dots n. \tag{3.4}$$

THEOREM 6. *Assume* $F_{\bar{i}} = G_{\bar{i}}$ *for all* $i = 1, \dots n$. *Then*
(i)

$$X \leqslant_{m3} Y \Leftrightarrow \begin{cases} Y \leqslant_{m1} X & \text{if } n \text{ is even,} \\ X \leqslant_{m1} Y & \text{if } n \text{ is odd;} \end{cases}$$

(ii) *if* n *is even,* $X \leqslant_{m3} Y$ *and* $X \leqslant_{m1} Y \Rightarrow F = G$;
(iii) *if* n *is odd,* $X \leqslant_{m3} Y$ *and* $Y \leqslant_{m1} X \Rightarrow F = G$.

PROOF. By Silvester's formula, for any $a \in \mathbb{R}^n$

$$P(X > a) = 1 - \sum_i P(X_i \leqslant a_i) + \sum_{i \neq j} P(X_i \leqslant a_i, X_j \leqslant a_j) + \dots (-1)^n P(X \leqslant a). \tag{3.5}$$

As the $(n-1)$-dimensional marginals coincide, all marginals of lower dimension coincide, too; therefore $X \leqslant_{m3} Y \Leftrightarrow P(X > a) \leqslant P(Y > a)$ for all $a \Leftrightarrow (-1)^n P(X \leqslant a) \leqslant (-1)^n P(Y \leqslant a)$ for all a. The rest is obvious. Q.E.D.

In the remainder of this section we depart from utility independent preferences and ask: Provided $X \leqslant_{m1} Y$, are there utility functions u^* other than those in U_{m1} for which $\Delta u^* \leqslant 0$ holds; in other words, is there a class U_{m1}^* larger than U_{m1} such that $\leqslant_{m1}$ means SD with respect to U_{m1}^*? The following Theorem 7 (the necessity part of which is due to Hadar and Russell 1974, Theorem 5.8) gives an answer in case X and Y are continuously distributed.

THEOREM 7. *If* X *and* Y *are continuously distributed,* $X \leqslant_{m1} Y \Leftrightarrow \Delta u \leqslant 0$ *for all*

$u \in U^*_{m1}$ where[3] *$U^*_{m1} := \{u \in \tilde{C}^n \mid u$ bounded above, $(-1)^{|J|+1}\partial u/\partial x_J \geq 0$ for all $j \subset N\}$, $\tilde{C}^n := \{u : \mathbb{R}^n \to \mathbb{R} \mid u$ n-times continuously differentiable outside finitely many hyperplanes$\}$.*

Note the inclusion $U_{m1} \cap \tilde{C}^n \subset U^*_{m1}$. The proof of sufficiency in Theorem 7 parallels that of Theorem 2. There are similar counterparts to Theorems 3, 4, and 5:

THEOREM 7A. *If X and Y are continuously distributed; $j \in \{2,3,4\}$, then $X \leq_{mj} Y$ $\Leftrightarrow \Delta u \leq 0$ for all $u \in U^*_{mj}$ where*
$U^*_{m2} := \{u \in U_0 \cap \tilde{C}^{2n} \mid u$ bounded above, $\partial^2 u/\partial x_J^2 \leq 0$ for all $J \subset N\}$,
$U^*_{m3} := \{u \in \tilde{C}^n \mid u$ bounded below, $\partial u/\partial x_J \geq 0$ for all $J \subset N\}$,
$U^*_{m4} := \{u \in U_0 \cap \tilde{C}^{2n} \mid u$ bounded below, $\partial^2 u/\partial x_J^2 \geq 0$ for all $J \subset N\}$.

The proof, again, is omitted.

To give an example, let A be a nonnegative matrix with n columns and m nonidentical rows, and let the function $u : \mathbb{R}^n \to \mathbb{R}$ be given by $u(x) = \max Ax$ if $x \geq 0$, $u(x) = 0$ else. Then u is in U^*_{m4}, while, in general, u is not in U_{m4}.

4. Proofs of Theorems 2 to 5

Theorem 2 can be proved as an application of Theorem 3(a) in Rüschendorf (1980). We will give a different proof here.

PROOF OF THEOREM 2. "$\Rightarrow$". Let $z = (z_1, z_2, \ldots z_n) \in \mathbb{R}^n$ be given. For $i = 1, 2, \ldots n$, $x_i \in \mathbb{R}$ define $u^i(x_i) = -1$ if $x_i \leq z_i$, $u^i(x_i) = 0$ else. Then u^i is ≤ 0 and nondecreasing, hence

$$-P(X \leq z) = E\left(-\prod_{i=1}^{n}\left(-u^i(X_i)\right)\right) \leq E\left(-\prod_{i=1}^{n}\left(-u^i(Y_i)\right)\right) = -P(Y \leq z).$$

"$\Leftarrow$". Let $u \in U_{m1}$, $u(x) = -\prod_{i=1}^{n} v^i(x_i)$ for all $(x_1, x_2, \ldots x_n) = x$ where each v^i is ≥ 0 and nonincreasing. First, we assume each v^i to be left continuous and $v^i(\infty) = 0$. Then measures μ_i on the reals are given by $\mu_i(]x_i, \infty[) := v^i(x_i)$, $x_i \in \mathbb{R}$. Define $\mu := \mu_1 \otimes \mu_2 \otimes \ldots \otimes \mu_n$ and, for every $x, z \in \mathbb{R}^n$, $g(z,x) = -1$ if $x < z$, $g(z,x) = 0$ else. Then

$$u(x) = -\mu(R_x) = \int_{\mathbb{R}^n} g(z,x)\,d\mu(z),$$

$$Eu(X) = \int_{\mathbb{R}^n} Eg(z,X)\,d\mu(z), \qquad Eu(Y) = \int_{\mathbb{R}^n} Eg(z,Y)\,d\mu(z).$$

Further, $P(X \leq z) \geq P(Y \leq z)$ for all z implies $Eg(z,X) = -P(X < z) \leq -P(Y < z) = Eg(z,Y)$ for all z; hence $Eu(X) \leq Eu(Y)$. Next, we allow for $v^i(\infty)$ being nonzero, $i = 1, 2, \ldots n$. Define $\alpha_i := v^i(\infty) \geq 0$, $\tilde{v}^i := v^i - \alpha_i$. Then

$$-u = \prod_{i \in N}(\tilde{v}^i + \alpha_i) = \sum_{J \subset N} \prod_{j \in J} \alpha_j \prod_{i \notin J} \tilde{v}^i,$$

$$Eu(X) - Eu(Y) = \sum_{J \subset N} \prod_{j \in J} \alpha_j \left[E \prod_{i \notin J} \tilde{v}^i(Y) - E \prod_{i \notin J} \tilde{v}^i(X)\right].$$

Note that the property $P(X \leq z) \geq P(Y \leq z)$ for all z implies the analogous property for each pair of marginal distributions of X and Y. Therefore, and as $\tilde{v}^i(\infty) = 0$ for all i, the term in square brackets is nonnegative for each J, hence $Eu(X) \leq Eu(Y)$. Finally, when some v^i are not left continuous, they can be approximated pointwise

[3] When $J = \{j_1, \ldots j_k\} \subset N$, denote $|J| := k$, $\partial u/\partial x_J := \partial^k u/(\partial x_{j_1} \ldots \partial x_{j_k})$, $\partial^2 u/\partial x_J^2 := \partial^{2k} u/(\partial x_{j_1}^2 \ldots \partial x_{j_k}^2)$. When $J = \emptyset$ define $\partial u/\partial x_J := u$.

from below by left continuous nonincreasing functions $v^{im} \geqslant 0$, $m \in \mathbb{N}$. $0 \leqslant v^{im} \nearrow v^i$ implies $0 \leqslant \prod v^{im} \nearrow \prod v^i$, and $\Delta u = \lim \Delta(-\prod v^{im}) \leqslant 0$.

Theorem 3 follows from Theorem 2 as it states a proposition dual to that of Theorem 2:

It can be easily seen that $U_{m3} = \{v \mid v(x) = -u(-x),\ u \in U_{m1}\}$, hence,

$$X \leqslant_{m3} Y \Leftrightarrow -Y \leqslant_{m1} -X \Leftrightarrow P(-Y \leqslant x) \geqslant P(-X \leqslant x) \quad \text{for all } x$$

$$\Leftrightarrow P(X \geqslant -x) \leqslant P(Y \geqslant -x) \quad \text{for all } x$$

$$\Leftrightarrow P(X > z) \leqslant P(Y > z) \quad \text{for all } z.$$

In the same way, Theorem 5 is deduced from Theorem 4. In order to prove Theorem 4 we need the following

LEMMA 2. *Let $k \in \{1,2,\ldots n\}$ and $a_1, a_2, \ldots a_k, x_{k+1} x_{k+2}, \ldots x_n \in \mathbb{R}$ be given. Then*

$$\int_{-\infty}^{a_k} \cdots \int_{-\infty}^{a_1} \prod_{i=1}^{k} (a_i - x_i) F(dx_1, \ldots dx_k, x_{k+1}, \ldots x_n)$$

$$= \int_{-\infty}^{a_k} \cdots \int_{-\infty}^{a_1} P(X_i \leqslant x_i \text{ for all } i = 1,2,\ldots n)\, dx_1 \ldots dx_k .$$

PROOF. For any given real numbers $a_1, \ldots a_k, x_{k+1}, \ldots x_n$ define auxiliary functions R^i, $i = 0,1,2,\ldots k$, which depend on $x^i := (x_1, x_2, \ldots x_i) \in \mathbb{R}^i$:

$$R^k(x^k) := F(x^k, x_{k+1}, \ldots x_n) = P(X_i \leqslant x_i \text{ for all } i = 1,2,\ldots n), \tag{4.1}$$

$$R^{i-1}(x^{i-1}) := \int_{-\infty}^{a_i} R^i(x^i)\, dx_i \tag{4.2}$$

for $i = 1,2,\ldots k$. Partial integration of (4.2) yields

$$R^{i-1}(x^{i-1}) = \int_{-\infty}^{a_i} (a_i - x_i) R^i(x^{i-1}, dx_i). \tag{4.3}$$

By backward recursion we get from (4.2) and (4.3) the formulae

$$R^0 = \int_{-\infty}^{a_k} \cdots \int_{-\infty}^{a_1} R^k(x^k)\, dx_1 \ldots dx_k \quad \text{and}$$

$$R^0 = \int_{-\infty}^{a_k} \cdots \int_{-\infty}^{a_1} \prod_{i=1}^{k} (a_i - x_i) R^k(dx_1, \ldots dx_k),$$

respectively; inserting (4.1) proves the lemma.

PROOF OF THEOREM 4. "$\Rightarrow$". Let $a \in \mathbb{R}^n$ and for $i \in N$, $x_i \in \mathbb{R}$, $u^i(x_i) := (x_i - a_i)_- := \min\{x_i - a_i, 0\}$. As the functions u^i are nondecreasing, concave and $\leqslant 0$ there holds $\Delta(-\prod_{i=1}^n(-u_i)) \leqslant 0$ by assumption. From this and from the lemma we conclude

$$\int_{L_a} P(X \leqslant x)\, dx = \int_{L_a} \prod_{i=1}^{n} (a_i - x_i) F(dx) = E \prod_{i=1}^{n} \left(-u^i(X_i)\right)$$

$$\geqslant E \prod_{i=1}^{n} \left(-u^i(Y_i)\right) = \int_{L_a} P(Y \leqslant y)\, dy.$$

"$\Leftarrow$". Let $u \in U_{m2}$; hence $-u(x) = \prod_{i=1}^n(-u^i(x_i))$ where $u^i \leqslant 0$ nondecreasing and concave. For every i, then, $u^i(\infty)$ is finite, and u^i can be written in the form $-u^i(x_i) = -u^i(\infty) + \int_{\mathbb{R}} (t_i - x_i)_-\, dv^i(t_i)$ where v^i is a nondecreasing, nonpositive function $\mathbb{R} \to \mathbb{R}$. We then have

$$-u(x) = \prod_{i\in N}\left(-u^i(x_i)\right) = \sum_{J\subset N}\prod_{i\notin J}\left(-u^i(\infty)\right)\prod_{i\in J}\int_{\mathbb{R}}(t_i - x_i)_-\,dv^i(t_i)$$

$$= \sum_{J\subset N}\alpha_J\int_{R^{|J|}}\prod_{i\in J}(t_i - x_i)_-\,d\Big(\prod_{i\in J}v^i(t_i)\Big)$$

with $\alpha_J := \prod_{i\notin J}(-u^i(\infty)) \geqslant 0$.

$$-Eu(X) = \sum_{J\subset N}\alpha_J\int_{\mathbb{R}^{|J|}}E\Big(\prod_{i\in J}(t_i - X_i)_-\Big)d\Big(\prod_{i\in J}v^i(t_i)\Big).$$

For every $J \subset N$ we get from Lemma 2 (applied to the distribution function $\tilde{F}_J$ of X_J)

$$E\prod_{i\in J}(t_i - X_i)_- = \int_{-\infty}^{t_J}\prod_{i\in J}(t_i - x_i)\tilde{F}_J(dx_J)$$

$$= \int_{-\infty}^{t_J}P(X_i \leqslant x_i \text{ for all } i \in J)\,dx_J. \tag{4.4}$$

Therefore, $-Eu(X)$ is a positively weighted sum of integrals with integrands of type (4.4). As an analogous representation holds for $-Eu(Y)$ we conclude from the assumption and from Lemma 1 that the inequality $-Eu(X) + Eu(Y) \geqslant 0$ must hold. Q.E.D.

5. The Multilinear Case

We recall the definitions of §1 and consider nondecreasing and bounded utilities of a more general type than so far. When each attribute A_i is utility independent of its complement $\mathscr{A}\setminus\{A_i\}$, $i = 1, \ldots n$, then u is *multilinear*, i.e. can be written (cf. Keeney and Raiffa 1976, p. 293)

$$u(x) = \sum_{I\subset N}\alpha^I\prod_{i\in I}u^i(x_i) \tag{5.1}$$

where the summation extends over all subsets $I \subset N$ and $\alpha^I \in \mathbb{R}$ are multi-indexed constants. Additionally, we may assume for all i

$$u^i \text{ nondecreasing and not constant,} \tag{5.2}$$

$$u^i(x_i) = 0 \qquad \text{for some} \quad x_i \in \mathbb{R} \cup \{-\infty, \infty\}. \tag{5.3}$$

(If not, we could substitute u^i by $-u^i$ and/or by $u^i - u^i(\infty)$ or $u^i - u^i(-\infty)$, respectively, which yielded a positive-affine transformation of u.) Note that any u^i may have positive and negative values.

The constants α^I may be positive or not. E.g., $\alpha^{12} > 0$ ($\alpha^{12} < 0$) means that an increase in x_1 yields a larger utility increase if x_2 is at a high (low) level than if x_2 is at a low (high) level. Lemma 3 gives sufficient conditions for $\alpha^I \geqslant 0$ and $\alpha^I \leqslant 0$.

Lemma 3. *Let u be multilinear with* (5.2) *and* (5.3) *and let all u^i be differentiable. For any nonvoid $I \subset N$:*

$$\frac{\partial u}{\partial x_I}(x)\left\{\begin{matrix}\geqslant\\ \leqslant\end{matrix}\right\}0 \qquad \textit{for all} \quad x \Rightarrow \alpha^I\left\{\begin{matrix}\geqslant\\ \leqslant\end{matrix}\right\}0.$$

Proof. For $I \subset N$, $I \neq \emptyset$

$$\frac{\partial u}{\partial x_I}(x) = \prod_{i\in I}u^{i\prime}(x_i)\cdot\sum_{I\subset J\subset N}\Big[\alpha^J\prod_{j\in J\setminus I}u^j(x_j)\Big]. \tag{5.4}$$

By (5.3) for $j \in N \setminus I$ there exist $x_j^0 \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $u^j(x_j^0) = 0$, hence in (5.4) all summands vanish where $J \neq I$:

$$\frac{\partial u}{\partial x_I}(x_I, x^0_{N\setminus I}) = \alpha^I \prod_{i\in I} u^{i'}(x_i).$$

(If some x_j^0 are infinite the left-hand side represents a limit.) For any $i \in I$, as u^i is not constant, there exists x_i^0 with $u^{i'}(x_i^0) > 0$; we conclude the lemma.

Whether $\partial u(x)/\partial x_I \geqslant 0$ for all x, or not, can (at least in principle) be checked by comparisons of lotteries: For given $I \subset N$ with $|I| = m$, given $a^0, a^1 \in \mathbb{R}^m$ with $a^0 \leqslant a^1$, and given $x_{N\setminus I}$ we have the simple relationship

$$\int_{a^0}^{a_1} \frac{\partial u}{\partial x_I} dx_I = \left[\sum_{\delta_i} (-1)^{m-\Sigma\delta_i} u(x)\right]_{x_i = a_i^{\delta_i}} = (Eu(M_1) - Eu(M_2))2^{m-1} \tag{5.5}$$

where the summation extends over all values of $\delta_i \in \{0,1\}$, $i = 1, \dots n$; $M_1 = M_1(I, a^0, a^1, x_{N\setminus I})$ denotes a lottery which gives payoffs in the set

$$\left\{x \in \mathbb{R}^n \mid x_i = a_i^{\delta_i}, \delta_i \in \{0,1\}, i \in I, m - \textstyle\sum \delta_i = 0, 2, \dots\right\}$$

each with equal probability 2^{1-m}. M_2 is defined analogously with payoff set

$$\left\{x \in \mathbb{R}^n \mid x_i = a_i^{\delta_i}, \delta_i \in \{0,1\}, i \in I, m - \textstyle\sum \delta_i = 1, 3, \dots\right\}.$$

(5.5) tells that $\partial u(x)/\partial x_I \geqslant 0$ ($\leqslant 0$) for all x if $M_1 \succcurlyeq M_2$ ($M_1 \preccurlyeq M_2$) for all a^0, a^1, and $x_{N\setminus I}$.[4]

Now, we turn to stochastic dominance with respect to the set

$$U_\delta := \left\{u \in U_0 \mid u \text{ multilinear}, u^i \in U^i; \text{ for } I \neq \emptyset, \alpha^I = 0 \text{ if } \delta^I = 0, \alpha^I\delta^I \geqslant 0 \text{ if } \delta^I \neq 0\right\}$$

for given U^i, $i = 1, \dots n$, and for a given *substitutional structure* δ, $\delta = (\delta^I)$, $\emptyset \neq I \subset N$, $\delta^I \in \{-1, 0, 1\}$, i.e. SD w.r.t. all multilinear utilities which have common signs of α^I and where u^i are taken from given classes U^i.

THEOREM 8. *$X \leqslant_{U_\delta} Y$ if and only if $\delta^I \cdot \Delta(\prod_{i\in I} u^i) \leqslant 0$ for all $u^i \in U^i$, $I \subset N$.*

PROOF. For any $u \in U_\delta$, $u = \sum \alpha^I \cdot \prod u^i$, holds $\Delta u = \sum |\alpha^I| \delta^I \Delta(\prod u^i)$ which proves sufficiency. To show necessity consider $I \subset N$ and $u^i \in U^i$; then $u := \delta^I \prod_{i\in I} u^i \in U_\delta$, and $0 \geqslant \Delta u = \delta^I \Delta(\prod u^i)$.

If a substitutional structure is given Theorem 8 reduces the efficiency analysis of multilinear utilities to that of multiplicative utilities for which criteria like those developed in §3 apply.

If there is no substitutional structure fully known one might consider SD with respect to U_δ where the sign restrictions on some α^J are relaxed. Let $J \subset N$ and α^J have positive as well as negative values; then $X \leqslant_{U_\delta} Y$ implies $\Delta(\prod_{i\in J} u^i) = 0$ for all $u^i \in U^i$, $i \in J$. If the classes U^i are rich enough[5] there follows that X_J and Y_J have identical probability distributions.

6. Conclusions

Stochastic dominance with respect to additive, multiplicative, and multilinear utilities has been investigated and simple criteria have been given. The above analysis can be extended into several directions. Multivariate analogs to univariate higher degree SD are along the lines of Theorems 4 and 5. Criteria for SD with utilities of type (1.3) and (1.4) where x^i are vectors can be developed similarly.

[4] If n is large, of course, this is no practical device. Compare Keeney and Raiffa (1976, pp. 297 ff) where assessing multilinear utilities is abandoned when $n > 4$.

[5] E.g., if they contain the increasing concave functions or the exponential/linear (cf. Rothblum 1975) functions.

As Theorems 7 and 7A show, the criteria derived under assumptions of utility independence and given substitutional structure carry over to more general situations where utility independence disappears and only the signs of certain partial derivatives remain given (which on their part determine the substitutional structure). Similar propositions hold in the multilinear case. Without assumptions of this type the results become much more involved and difficult to implement; for those results the reader is referred to Mosler (1982).

Finally, let us compare the SD approach with existing utility function methods in multiattribute decision making. A direct method most propagated (cf. Keeney and Raiffa 1976) for real life applications proceeds as follows: first, assess the probability distributions of the consequences; second, state and validate assumptions on utility independence of the DM's preferences; third, assess n univariate utilities and (up to) $2^n - 2$ scaling constants; fourth, evaluate expected utilities. When using the SD approach the first and the second step are the same while in Step 3 only the signs of the scaling constants are to be assessed (and, possibly, qualitative properties such as univariate risk aversion). In Step 4 instead of expected utilities a SD criterion is to be evaluated. If one alternative comes out to be dominant over the other the SD approach needs a considerably smaller amount of data than the direct approach; if neither alternative is dominant Steps 3 and 4 of the direct approach have to be carried out.

More generally, when being applied to a given set of consequences every SD rule yields an efficient set of those consequences which are not dominated with respect to the class of utilities considered; i.e., all inferior consequences are excluded. A two-stage decision procedure seems to be natural: in a first stage an appropriate SD rule and its proper efficient set are determined, and, in a second stage, a final decision is chosen from the efficient set by means of an arbitrary multivariate decision procedure.

In most practical situations, of course, the set of consequences is not given a priori, and quite often it seems to be more difficult to obtain detailed information on the probability distribution functions of the consequences than to assess the utility function. In these cases, first, qualitative properties of the utility function may be assessed; then, the proper SD rule yields inequalities for the probabilities of the consequences. So, the probabilities of inferior consequences need not be specified completely, but only bounds of these probabilities must be assessed.[6]

[6]The author is indebted to an anonymous referee for his valuable suggestions concerning the title and the conclusions of this paper.

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